UNIT 12

# Trigonometry

# Introduction

In Unit 8 you met the idea of similar triangles, that is, triangles that are the same shape but not necessarily the same size. For example, similar triangles were used in Unit 8, Section 3, to find the height of a tree by using the length of its shadow.

Methods of using triangles to find unknown lengths and unknown angles have a long history, and the branch of mathematics that is concerned with such methods is called **trigonometry**.

The word trigonometry derives from the Greek words *trigon* for triangle and *metron* for measure.

Section 1 introduces the basic ideas of trigonometry, including the *sine*, *cosine* and *tangent* of an angle. Then these ideas are applied to problems that involve calculating unknown side lengths and unknown angles in right-angled triangles, a procedure known as *solving the triangle*.

Section 2 looks at the problem of solving triangles that do not necessarily have a right angle. This procedure has many applications in navigation, surveying and astronomy, and some of these applications are explored in this section, including estimating the speed of a glacier. Also in Section 2 the problem of finding the area of a triangle is revisited – how do you calculate the area if you know information other than the base and the height?

Section 3 is a change of direction. It turns out that the mathematics developed initially to solve triangles is also very useful in describing circular motion and other repetitive phenomena. Section 3 looks at this wider view of the use of trigonometry.

Finally, Section 4 introduces radians as an alternative to degrees as a measure of angles. You will see that, in a sense, radians are a more natural way to measure angles than degrees, and for this reason they are usually used in higher-level mathematics modules.

Radians are the default angle measure on many calculators and spreadsheets.

The MU123 Guide is needed for three of the activities in this unit.

Activities 3, 8 and 32 on pages 63, 69 and 107, respectively, are in Section 3 of the MU123 Guide.

# 1 Right-angled triangles

This section introduces the trigonometric ratios sine, cosine and tangent, and explains how they can be used in practical situations to solve geometric problems. Subsection 1.1 explains how the trigonometric ratios are defined and how you can find them using your calculator. Subsections 1.2 and 1.3 show you how these ratios can be used to find unknown lengths and unknown angles in right-angled triangles. In the final subsection you will see that for some special angles the trigonometric ratios can be calculated directly without using your calculator.

## 1.1 Sine, cosine and tangent

Let's start with a simple example of the kind of problem that can be solved by using trigonometry. In Unit 8 you saw a method for finding the height of a tall tree, which involved measuring the length of the shadow of the tree, and comparing it to the length of the shadow of a stick. However, there is another method, which can be used even if the sun isn't shining!

The idea is to walk a known distance away from the tree, and measure the angle $\theta$ from where you are standing to the top of the tree, as shown in Figure 1. An angle of this type, measured upwards from the horizontal, is called an **angle of elevation**, and it can be found by using a device called a *clinometer*, shown in Figure 2.

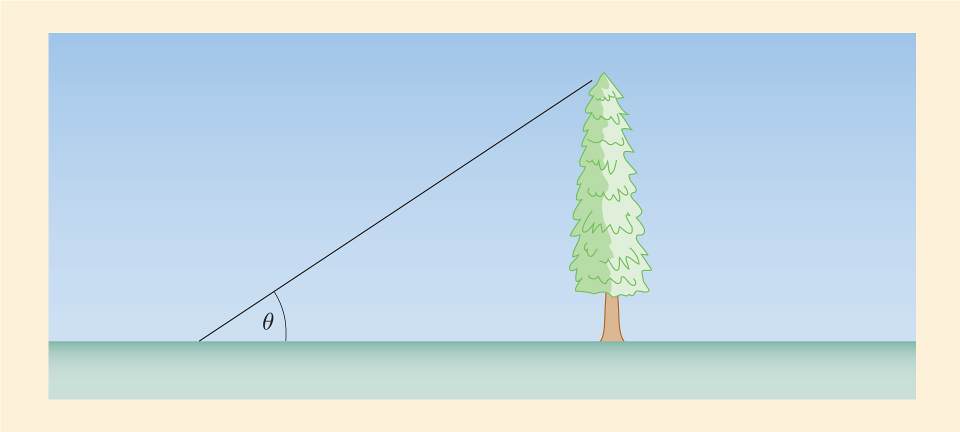



**Figure 1** The angle of elevation of a tree

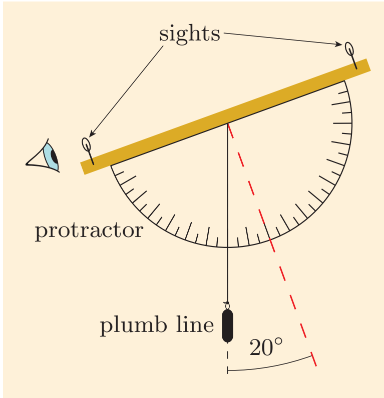



**Figure 2** A clinometer showing an angle of elevation of 20°

Suppose that you walk 100 metres away from a particular tree, and the angle of elevation of the top of the tree from this distance turns out to be 20°. The information that you know about the height of the tree is summarised in Figure 3. The vertical side of the right-angled triangle represents the tree, and the horizontal side represents the line between the foot of the tree and the point to which you walked. The lengths marked are in metres, with the height of the tree denoted by $x$.

For simplicity, it is assumed here that the angle is measured when the clinometer is on the ground and that the ground is horizontal.

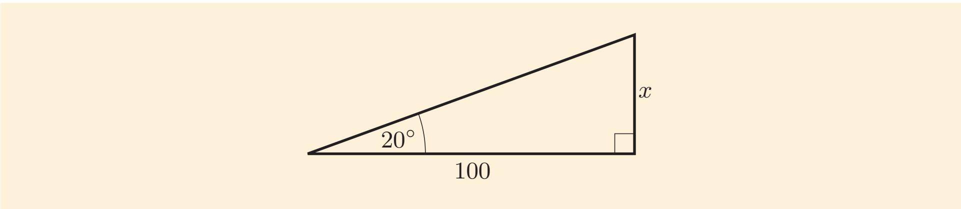



**Figure 3** A diagram showing the information known about the height of the tree

The height of the tree can be worked out from this diagram, as it contains enough information for there to be only one answer for $x$. But how do you find $x$? This is where trigonometry comes in – it's all about the relationships between lengths and angles.

So let's look at the basic ideas of trigonometry, and then we'll come back to the problem of finding the height of the tree, and use trigonometry to solve it.

Basic trigonometry is all about the relationships between the lengths of the sides and the angles within right-angled triangles. As an illustration, consider the right-angled triangle shown in Figure 4 (overleaf). It has two acute angles as well as the right angle, and we will choose to focus on just one of these acute angles, the one marked $\theta$ in the diagram.

Remember that an *acute* angle is one between 0° and 90° (exclusive).

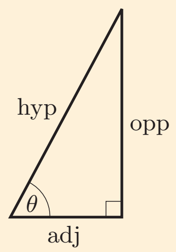



**Figure 4** A right-angled triangle with chosen angle $\theta$

One of the sides of the triangle is the hypotenuse (the side opposite the right angle), and you can distinguish the other two sides by using the fact that one of them is **opposite** the chosen angle $\theta$, while the other is **adjacent** to it. The sides of the triangle have been marked hyp, opp and adj accordingly, for 'hypotenuse', 'opposite' and 'adjacent'.

Now look at Figure 5, which shows three triangles similar to the triangle in Figure 4. So all four triangles have the same three angles. The sides of these triangles are also marked hyp, opp and adj in relation to the angle $\theta$.

Remember that *similar* triangles are the same shape but not necessarily the same size. So saying that two triangles are similar is the same as saying that one is a scaled (and possibly flipped) version of the other.

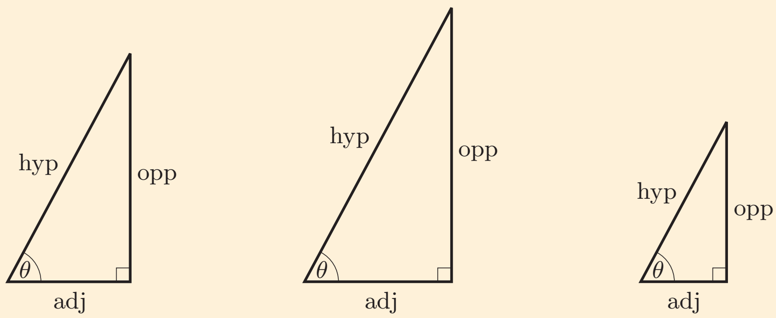



**Figure 5** Three right-angled triangles similar to the triangle in Figure 4

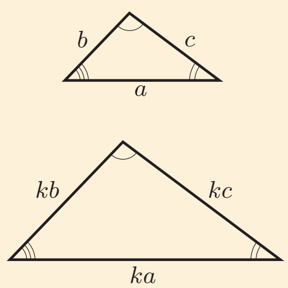



**Figure 6** Two similar triangles

The crucial fact which forms the foundation of trigonometry is that if you find the ratio of the lengths of two sides of a triangle and also the ratio of the lengths of the corresponding two sides of a similar triangle, then you get the same answer in each case. This fact applies to all similar triangles, not just right-angled ones, and it is illustrated by Figure 6. This figure shows a triangle with sides of lengths $a$, $b$ and $c$, and a second, scaled triangle in which the side lengths are multiplied by the scale factor $k$. The ratio of the left slant side to the right slant side in the first triangle is $b/c$, and the ratio of the corresponding two sides in the second triangle is $(kb)/(kc)$, which is equal to $b/c$. So scaling a triangle does not affect the ratio of the lengths of any two of its sides.

This means that for each of the triangles in Figures 4 and 5, the ratio of the lengths of the opposite and adjacent sides, that is

$$\frac{\text{opp}}{\text{adj}},$$

has the same value. And the same is true for the ratios of the lengths of the other pairs of sides,

$$\frac{\text{opp}}{\text{hyp}} \quad \text{and} \quad \frac{\text{adj}}{\text{hyp}}.$$

These three ratios are the key to solving problems involving the

relationships between lengths and angles, and they are given the special names below.

### *Trigonometric ratios*

Suppose that $\theta$ is an acute angle in a right-angled triangle in which the lengths of the hypotenuse, opposite and adjacent sides are represented by hyp, opp and adj, respectively, as in Figure 4.

The **sine** of the angle $\theta$ is

$$\sin\theta = \frac{\text{opp}}{\text{hyp}}.$$

The **cosine** of the angle $\theta$ is

$$\cos\theta = \frac{\text{adj}}{\text{hyp}}.$$

The **tangent** of the angle $\theta$ is

$$\tan\theta = \frac{\text{opp}}{\text{adj}}.$$

These expressions are read as:

sine *thee*-ta,
cos *thee*-ta,
tan *thee*-ta.

A popular method of remembering these definitions is to take the initial letters from

Sine = Opp/Hyp, Cosine = Adj/Hyp, Tangent = Opp/Adj,

to make the acronym

SOH CAH TOA.

This acronym is read to rhyme with Krakatoa.

This acronym tells you the sides used in each ratio, and which side is divided by which. For example, SOH tells you that to find the sine of an angle in a right-angled triangle, you divide the length of the opposite side by the length of the hypotenuse.

Remember that in practical problems the lengths must be measured in the same units.

The sine, cosine and tangent of an acute angle $\theta$ each depend only on the size of the angle $\theta$, and not on the particular right-angled triangle that $\theta$ is in – as you have seen, these values are the same for all such right-angled triangles. You can find the approximate values of the sine, cosine and tangent of any acute angle by drawing a suitable right-angled triangle and measuring its sides. For example, Figure 7 shows a right-angled triangle with an acute angle of 34°. This is an accurate scale drawing, and if you measure the sides then you will find that they are approximately 3.5 cm, 5.2 cm and 6.3 cm, as marked. The sides have also been marked opp, adj and hyp in relation to the angle of 34°.

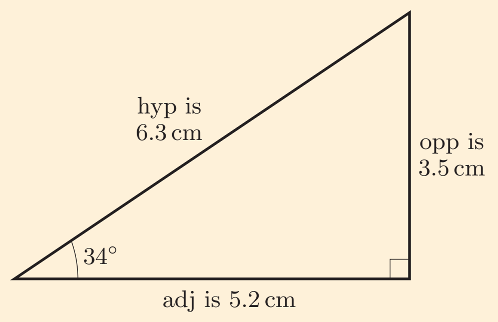



**Figure 7** A right-angled triangle with an angle of 34°

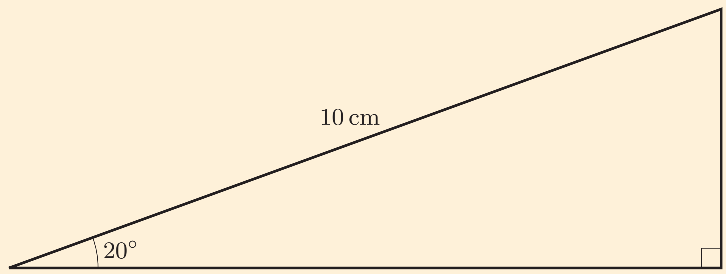
If you apply the definitions of sin, cos and tan to the triangle in Figure 7 (using the acronym SOH CAH TOA to help you remember them), then you obtain

$$\sin 34° = \frac{\text{opp}}{\text{hyp}} \approx \frac{3.5}{6.3} \approx 0.56,$$
$$\cos 34° = \frac{\text{adj}}{\text{hyp}} \approx \frac{5.2}{6.3} \approx 0.83,$$
$$\tan 34° = \frac{\text{opp}}{\text{adj}} \approx \frac{3.5}{5.2} \approx 0.67.$$

In the activity below, you are asked to find approximate values for the sine, cosine and tangent of the angle 20°.

## Activity 1 *Finding approximate values for trigonometric ratios*

The diagram below shows a right-angled triangle with an acute angle of 20° and one side of length 10 cm.




(a) Mark the sides of the triangle as opp, adj and hyp in relation to the angle 20°.

(b) By measuring the side lengths to the nearest millimetre, find the approximate values of $\sin 20°$, $\cos 20°$ and $\tan 20°$.

Because the longest side of any right-angled triangle is the hypotenuse, the opposite and adjacent side lengths are always less than the length of the hypotenuse. Thus the sine and cosine of an acute angle are always less than 1. However, the tangent of an acute angle can be less than 1 (as it is for 34° and 20°), or equal to 1, or greater than 1.

You are now ready to return to the problem of finding the height of the tree. The value of $\tan 20°$ that you found in Activity 1 can be used to solve this problem, at least approximately. The diagram giving the known information about the height of the tree is repeated in Figure 8, with the sides labelled opp, adj and hyp in relation to the angle of 20°. The lengths are in metres.

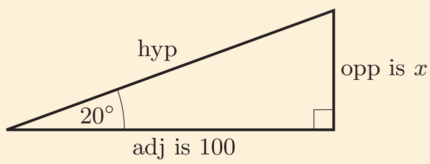



**Figure 8** The diagram for the tree height problem

From this diagram,

$$\tan 20^\circ = \frac{\text{opp}}{\text{adj}} = \frac{x}{100}.$$

But you already know the approximate value of $\tan 20^\circ$, from Activity 1. So

$$\frac{x}{100} \approx \text{the value for } \tan 20^\circ \text{ found in Activity 1.} \qquad (1)$$

In the activity below you are asked to use this equation to find the approximate height of the tree.

**Activity 2** *Finding an approximate answer to the tree height problem*

Use equation (1) and the approximate value for $\tan 20^\circ$ found in Activity 1 to find the approximate height of the tree.

Although approximate values for the trigonometric ratios of different angles can be obtained by drawing and measuring right-angled triangles, in practice it is much quicker to obtain accurate sine, cosine and tangent values from a calculator or computer. The next activity shows you how to do this.

**Activity 3** *Trigonometric ratios on your calculator*

This activity is in Subsection 3.7 of the MU123 Guide.

For centuries, values of sines, cosines and tangents of acute angles were available only in mathematical tables, such as those shown in Figure 9.

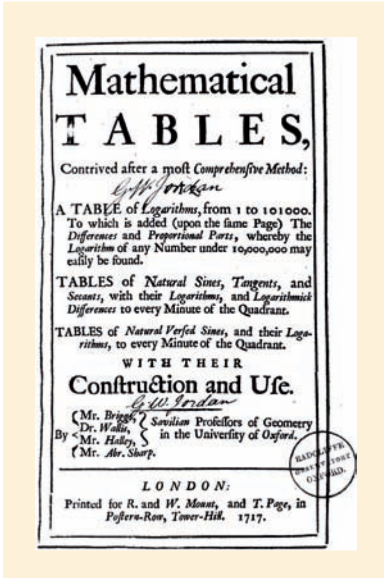

Mathematical
TABLES,
Contrived after a most Comprehensive Method:
A TABLE of Logarithms, from 1 to 101000. To which is added (upon the same Page) The Differences and Proportional Parts, whereby the Logarithm of any Number under 10,000,000 may easily be found.
TABLES of Natural Sines, Tangents, and Secants, with their Logarithms, and Logarithmick Differences to every Minute of the Quadrant.
TABLES of Natural Versed Sines, and their Logarithms, to every Minute of the Quadrant.
WITH THEIR
Construction and Use.
By Mr. Briggs, Dr. Wallis, Mr. Halley, Mr. Abr. Sharp. Savilian Professors of Geometry in the University of Oxford.
LONDON: Printed for R. and W. Mount, and T. Page, in Postern-Row, Tower-Hill. 1717.

**Figure 9** The title page of an eighteenth-century book of mathematical tables

Arguably the earliest table of trigonometric values is the table of lengths of chords in a circle in Claudius Ptolemy's *Almagest*, a book on mathematical astronomy, written in about AD 150 and known to us through later Arabic translations. In the *Almagest*, Ptolemy tabulated the lengths of the sides of many right-angled triangles, taking the hypotenuse to be of length 60.

Ptolemy's work became widely known in Europe after the publication in 1496 of the *Epitome of the Almagest*, by the German astronomers Georg Peurbach and Johannes Müller von Königsberg, known as Regiomontanus; for example, it was used by Copernicus and Galileo.

The first person to relate sines and cosines directly to angles in a triangle as we do today was the Austrian astronomer and mathematician Georg Joachim Rheticus, in his pamphlet *Canon doctrinae triangulorum* of 1551. Rheticus' masterwork, his immense *Opus palatinum de triangulis* of 1596, which ran to some 1500 pages, contained tables calculated to ten decimal places, which were of such accuracy that they were considered the standard until the early twentieth century.

The word *sine* has its origins in Sanskrit, coming to us through Arabic and Latin. The Latin word *sinus* appeared in European mathematics texts in the twelfth century.

In the next two subsections you will use sine, cosine and tangent values obtained from your calculator to solve problems involving right-angled triangles.

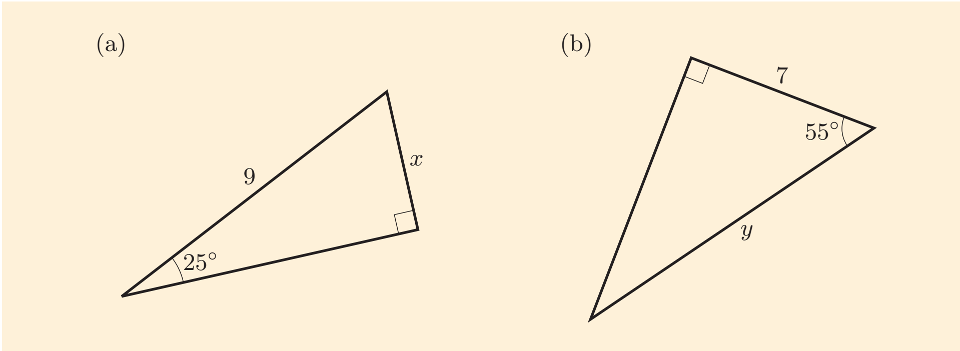
## 1.2 Finding unknown lengths

The following example shows you how to use the trigonometric ratios to find unknown lengths in two right-angled triangles.

Tutorial clip

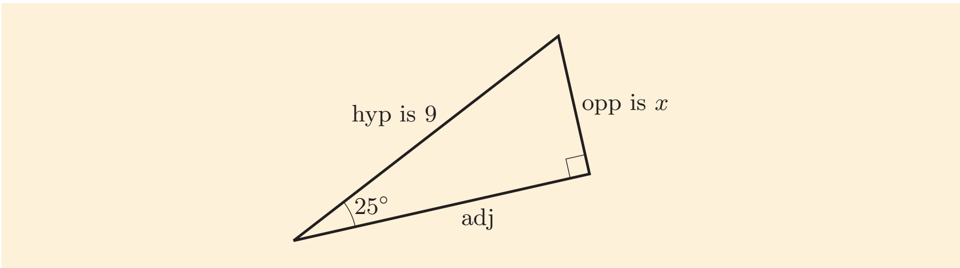
**Example 1** *Finding unknown lengths*

Find the lengths $x$ and $y$ in the triangles below, to three significant figures.




### Solution

(a) Label the sides of the triangle in relation to the given angle.




Use SOH CAH TOA. The unknown side $x$ is opposite the angle of 25°, and the known side is the hypotenuse, so the appropriate trigonometric ratio is sine.

From the diagram,

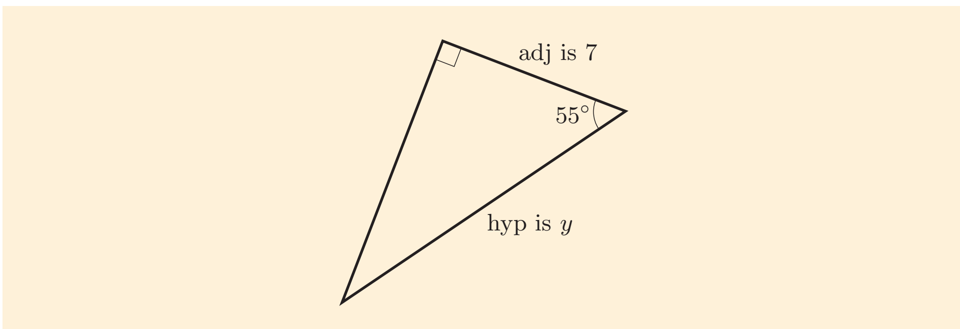
$$\sin 25° = \frac{\text{opp}}{\text{hyp}} = \frac{x}{9}, \quad \text{so} \quad x = 9\sin 25°.$$

A calculator gives $\sin 25° = 0.422\,618\ldots$, so

$$x = 9\sin 25° = 9 \times 0.422\ldots = 3.803\ldots.$$

Hence $x = 3.80$ (to 3 s.f.).

(b)



From the diagram,

$$\cos 55^\circ = \frac{\text{adj}}{\text{hyp}} = \frac{7}{y}, \quad \text{so} \quad y\cos 55^\circ = 7.$$

This gives

$$y = \frac{7}{\cos 55^\circ} = 12.204\ldots.$$

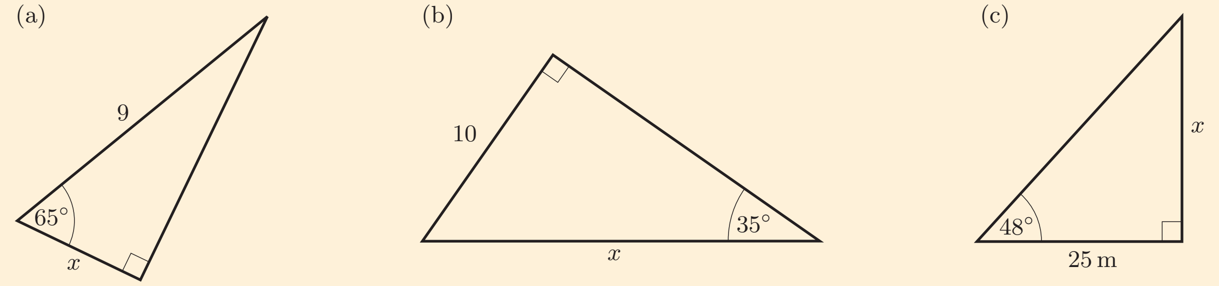
Hence $y = 12.2$ (to 3 s.f.).

As a check, notice that the value of $y$ is greater than 7, as you would expect since $y$ is the length of the hypotenuse.

Here are some similar questions for you to try. In each of them, you may find it helpful to draw your own diagram of the triangle, showing which sides are the opposite, adjacent and hypotenuse, as was done in Example 1.

## Activity 4 *Finding unknown lengths*

For each of the following triangles, calculate the unknown length $x$ to three significant figures.

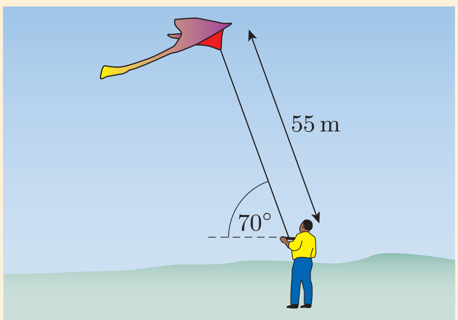



The triangle in part (a) is congruent to the triangle in Example 1(a). This illustrates that you can sometimes choose to use either sine or cosine to solve a problem.

In many geometric problems, drawing a clear diagram helps you to see how the problem can be solved. This is illustrated in the next example, which asks you to calculate the height of a kite above the ground.

## Example 2 *Finding the height of a kite*

A child is flying a kite and holds the end of the string at a height of 0.9 metres above the ground. The length of the string is 55 metres. The string is taut and makes an angle of 70° with the horizontal, as shown in the figure below.



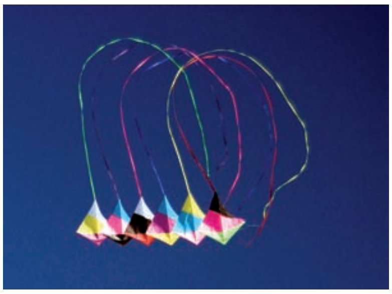



**Figure 10** Stunt kites

(a) Calculate the height of the kite above the child's hand.

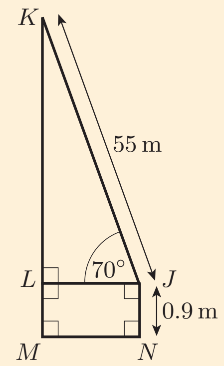
(b) Hence calculate the height of the kite above the ground.

Give your answers to the nearest 10 cm.

### Solution

Draw a diagram, mark the known measurements, and decide which lengths you need to find.

This diagram is not to scale.




(a) The height of the kite above the child's hand is represented by the line segment $KL$. Now $\triangle KJL$ is a right-angled triangle, in which

$$\sin 70^\circ = \frac{KL}{55}, \quad \text{so} \quad KL = 55 \sin 70^\circ = 51.68\ldots.$$

Thus the height of the kite above the child's hand is 51.7 m (to the nearest 10 cm).

(b) From the diagram, the height of the kite above the ground is $KM = KL + LM$. So, by part (a),

$$KM = 51.68\ldots + 0.9 = 52.58\ldots.$$

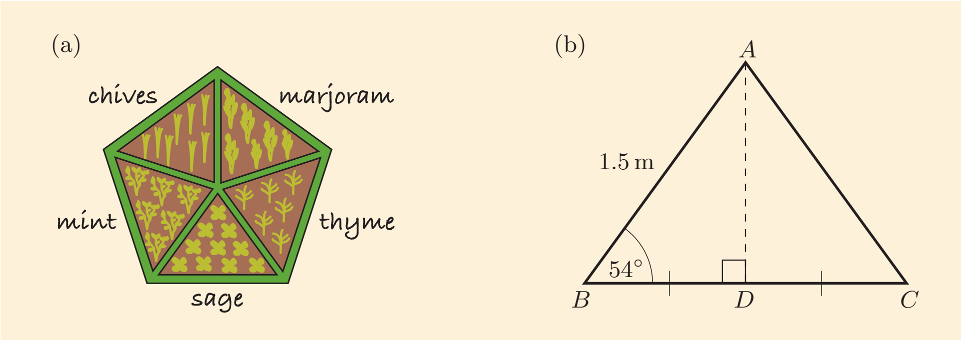
Therefore the height of the kite above the ground is 52.6 m (to the nearest 10 cm).

The trigonometric ratios apply only to right-angled triangles. So sometimes you need to add construction lines to your diagram to form right-angled triangles before using these ratios, as illustrated in the next activity.


## Activity 5 *Designing a herb garden*

Diagram (a) at the top of the next page shows a design for a herb garden in a wooden frame shaped like a regular pentagon. The struts from the centre of the pentagon to its vertices are each of length 1.5 m.

Diagram (b) shows one triangular section of the wooden frame with some of the measurements marked. Vertex $A$ of the triangle corresponds to the centre of the pentagon, and $B$ and $C$ are vertices of the pentagon. A construction line $AD$ has been added to split $\triangle ABC$ into two right-angled triangles.




From Unit 8, each interior angle of a regular pentagon is 108°, so $\angle ABD = \frac{1}{2} \times 108° = 54°$.

What length of wood is needed for the edge labelled $BC$, correct to the nearest millimetre?

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Trigonometric ratios can be useful in calculations involving the locations of objects such as ships or planes. Often the location of such an object is given by:

- the distance of the object from some known point;
- the direction of the object from the known point, measured as a **bearing**, that is, the number of degrees west or east of the north or south direction.

For example, Figure 11 shows a point $B$ specified as 5 km from a point $A$ in the direction 35° east of south of $A$. This direction is written concisely as S 35°E of $A$.

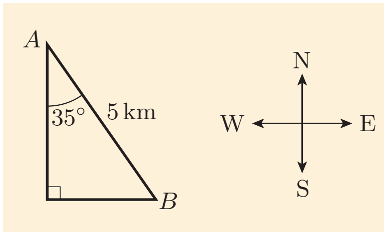



**Figure 11** A point $B$ that is 5 km in the direction 35° east of south of a point $A$

**Figure 12** The dial of a compass used for navigation

However, it can be more useful to specify how far north or south and how far east or west a point is from a known point. You can use the trigonometric ratios to calculate these distances, as in the following example.

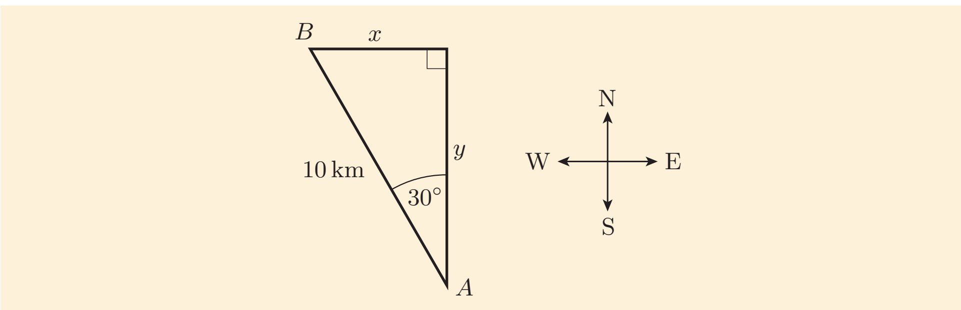
## Example 3 *Using bearings*

A ship is 10 km in the direction 30° west of north from a lighthouse. Calculate how many kilometres west and how many kilometres north the ship is from the lighthouse, correct to one decimal place.

### Solution

First draw a diagram and label the unknown lengths.

In the diagram, the point $A$ represents the lighthouse and the point $B$ represents the ship.




The hypotenuse of the right-angled triangle has length 10 km.

The distance $x$ km that $B$ lies to the west of $A$ is the length of the side opposite the angle of 30°. Thus

$$\sin 30^\circ = \frac{x}{10}, \quad \text{so} \quad x = 10 \sin 30^\circ = 5.$$

Hence the ship is 5.0 km west of the lighthouse (correct to 1 d.p.).

Similarly, the distance that $B$ lies to the north of $A$ is the length $y$ of the side adjacent to the angle of 30°. Thus

$$\cos 30^\circ = \frac{y}{10}, \quad \text{so} \quad y = 10 \cos 30^\circ = 8.66\ldots.$$

Hence the ship is 8.7 km north of the lighthouse (correct to 1 d.p.).

You can apply the same method to the following problem.

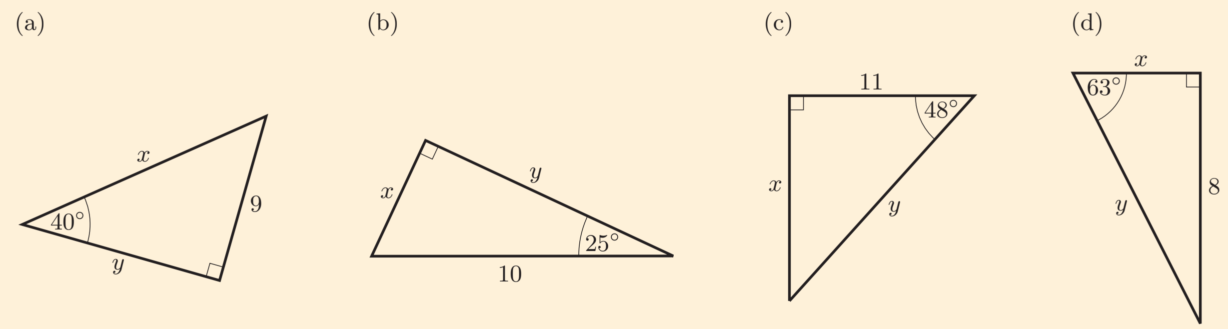
## Activity 6 *Using bearings*

A plane travels a distance of 100 km in the direction 20° east of north. Calculate how far north it is from its starting position, to the nearest kilometre.

The solutions to the activities and examples in this subsection have depended on choosing the most appropriate trigonometric ratio to use. The next activity gives you some further practice in identifying these ratios.

## Activity 7 *Choosing which trigonometric ratio to use*

For each of the following triangles, calculate the lengths $x$ and $y$ to three significant figures.




In each part of Activity 7, once you had found one of the unknown sides, you could have used Pythagoras' Theorem to find the other unknown side, instead of a second trigonometric ratio. However, it is usually better to use the values that you have been given, rather than values that you have calculated, in case you have made a numerical slip.

In this subsection you have seen how the three trigonometric ratios sine, cosine and tangent are defined in a right-angled triangle, and you have used these ratios to find unknown lengths in both practical and abstract

problems. In the next subsection you will see how to use these ratios to find unknown angles.

## 1.3 Finding unknown angles

Not all problems involving triangles are to find an unknown length. Sometimes you want to find the size of an angle. For example, consider the diagram in Figure 13.

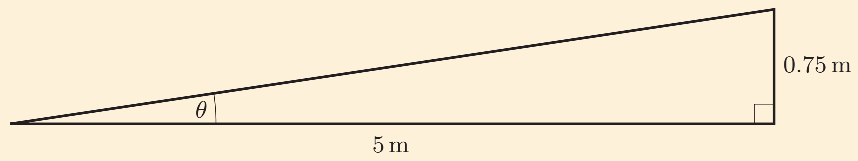



**Figure 13** The angle of a ramp

This shows a ramp that extends over a length of 5 m, reaching a height of 0.75 m at its raised end. Suppose that you want to know the size of the angle $\theta$, for example to determine whether the ramp is suitable for wheelchair users.

You can see from Figure 13 that the angle $\theta$ satisfies

$$\tan\theta = \frac{0.75}{5} = 0.15.$$

So the problem now is to find the angle $\theta$ whose tangent is 0.15.

If you know the tangent of an acute angle (or the sine or cosine of the angle), then you can use your calculator to find the angle. The notation used for the acute angle whose tangent is the number $x$ is

$$\tan^{-1}(x).$$

The expression $\tan^{-1}(x)$ is read as 'tan to the minus 1 of $x$' or as 'the inverse tangent of $x$'.

Similarly, the notations used for the acute angle whose sine is $x$ and for the acute angle whose cosine is $x$ are

$$\sin^{-1}(x) \quad \text{and} \quad \cos^{-1}(x),$$

respectively.

Note that $\tan^{-1}(x)$ is not the same as $(\tan x)^{-1}$, which equals $1/\tan x$, and similar comments apply to $\sin^{-1}(x)$ and $\cos^{-1}(x)$.

The angles $\sin^{-1}(x)$, $\cos^{-1}(x)$ and $\tan^{-1}(x)$ are called the **inverse sine**, **inverse cosine** and **inverse tangent** of $x$, respectively. You will see in the next activity how to use your calculator to find such angles. For the wheelchair ramp problem, a calculator gives

$$\theta = \tan^{-1}(0.15) = 8.5^\circ \text{ (to 2 s.f.)}.$$

The inverse sine, inverse cosine and inverse tangent of $x$ are also called the **arcsine**, **arccosine** and **arctangent** of $x$, with the alternative notations:

$$\arcsin(x) = \sin^{-1}(x),$$

$$\arccos(x) = \cos^{-1}(x)$$

and

$$\arctan(x) = \tan^{-1}(x).$$

So the ramp is at an angle of about 8.5°.

The next activity shows you how to find angles from trigonometric ratios using your calculator.

### Activity 8 *Finding angles from trigonometric ratios*

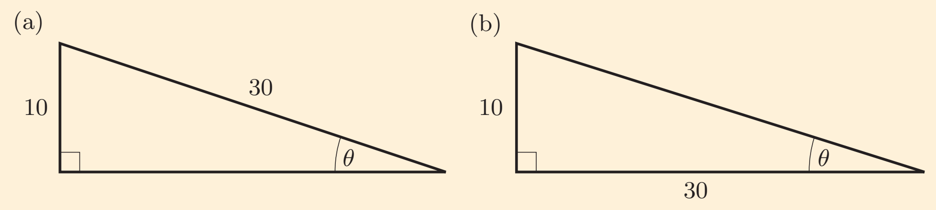
This activity is in Subsection 3.8 of the MU123 Guide.

The following example shows you how to find unknown angles in a right-angled triangle.

Tutorial clip

## Example 4 *Finding unknown angles*

For each of the triangles below, find the angle $\theta$ to the nearest degree.




### Solution

(a) Here you know the length of the hypotenuse and the length of the side opposite the unknown angle.

In this triangle,

$$\sin\theta = \frac{10}{30} = \frac{1}{3}.$$

So $\theta$ is the acute angle whose sine is $\frac{1}{3}$.

Hence

$$\theta = \sin^{-1}\left(\tfrac{1}{3}\right).$$

A calculator gives $\sin^{-1}\left(\frac{1}{3}\right) = 19.471\ldots^\circ$, so $\theta = 19^\circ$ (to the nearest degree).

(b) Here you know the length of the side opposite the unknown angle and the length of the side adjacent to this angle.

In this triangle,

$$\tan\theta = \frac{10}{30} = \frac{1}{3}.$$

Hence

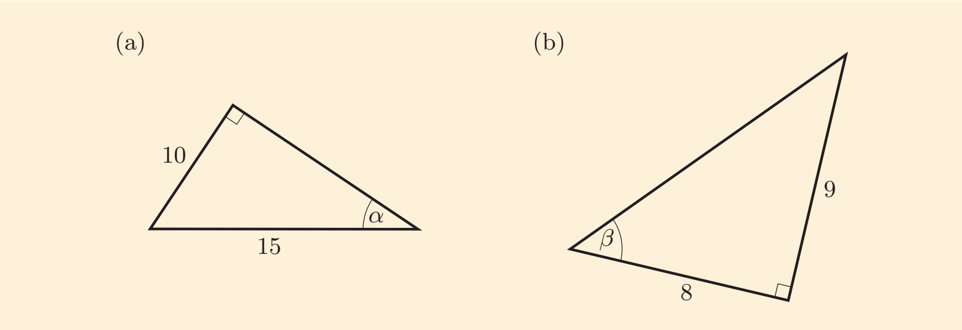
$$\theta = \tan^{-1}\left(\tfrac{1}{3}\right).$$

A calculator gives $\tan^{-1}\left(\frac{1}{3}\right) = 18.434\ldots^\circ$, so $\theta = 18^\circ$ (to the nearest degree).

Here are some similar questions for you to try.

## Activity 9 *Finding unknown angles*

For each of the following triangles, find the labelled angle to the nearest degree.




Practical problems often require you to use some of the geometric results that you met in Unit 8, as well as the trigonometric ratios, as the next example shows. This example also illustrates that it is important to convert all length measurements to the same unit before using trigonometric ratios.

## Example 5 *Finding a glide angle*

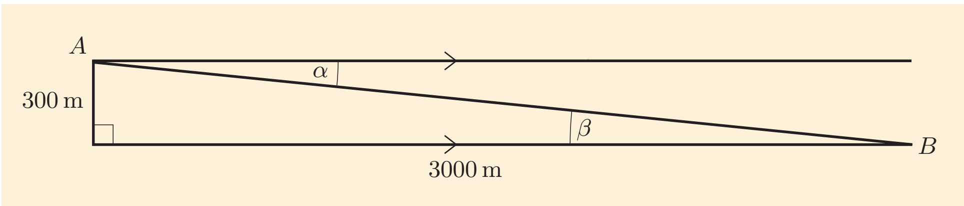
In a hang-gliding competition, participants have to fly a horizontal distance of 3 km from a height of 300 m and land on a target. The glide angle is the angle between the horizontal and the downward path of the glider. If it is assumed that the glide angle is constant throughout the flight, what angle is required if the participant is to land at the target?

**Figure 14** Hang gliding

### Solution

Convert the length measurements to the same unit and mark these on a clear diagram. Give the unknown angle a variable name. Do the same for other angles, where this is helpful.

In the diagram, the glider takes off from $A$ and aims to land at $B$. The horizontal distance to be travelled is 3000 m and the vertical distance to descend is 300 m. The required glide angle is denoted by $\alpha$ and the angle at $B$ is denoted by $\beta$.




Use geometric results and trigonometry to find the required angle.

The angles $\alpha$ and $\beta$ are equal, since they are alternate angles between two horizontal lines.

From the diagram, $\tan\beta = \dfrac{\text{opp}}{\text{adj}} = \dfrac{300}{3000} = 0.1$.

Hence $\beta = \tan^{-1}(0.1) = 5.71\ldots°$.

So the required glide angle $\alpha$ is about 6°.

The next activity also involves calculating an angle, and then interpreting the result.

**Figure 15** An avalanche on a snowy mountainside

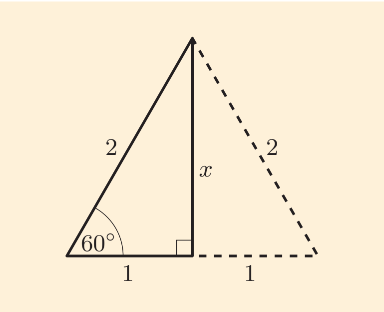



**Figure 16** Dividing an equilateral triangle in half

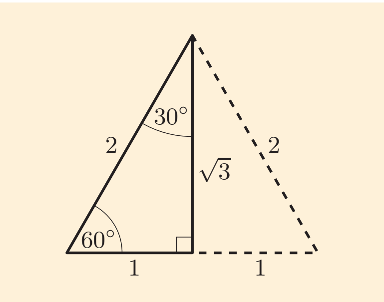



**Figure 17** Height of an equilateral triangle

**Activity 10** *Checking for an avalanche*

From a map, a skier knows that a hillside rises steadily through a vertical height of 150 m over a horizontal distance of 175 m. There is an avalanche risk if the slope of the snow is between 35° and 45°.

If it is assumed that the hillside is covered with a uniform layer of snow, is there an avalanche risk?

## 1.4 Useful trigonometric ratios and identities

In this subsection, you will see how the trigonometric ratios for the angles 30°, 45° and 60° can be worked out directly, without using your calculator. These values are quite memorable and useful to know. They can be calculated from triangles in which these angles occur.

For example, in an equilateral triangle, the interior angles are each 60°. Figure 16 shows an equilateral triangle with sides of length 2 units, in which a vertical line divides the base of the triangle into two equal parts, each of length 1. (Choosing the equilateral triangle to have sides of length 2 makes the calculations easier.)

From the right-angled triangle on the left-hand side of the equilateral triangle you can see that

$$\cos 60° = \frac{1}{2}.$$

The length, $x$, of the third side of this right-angled triangle can be calculated by using Pythagoras' Theorem:

$$1^2 + x^2 = 2^2, \quad \text{so} \quad x^2 = 4 - 1 = 3.$$

Hence $x = \sqrt{3}$ units, as shown in Figure 17. Since the side opposite the angle 60° is of length $\sqrt{3}$,

$$\sin 60° = \frac{\sqrt{3}}{2} \quad \text{and} \quad \tan 60° = \frac{\sqrt{3}}{1} = \sqrt{3}.$$

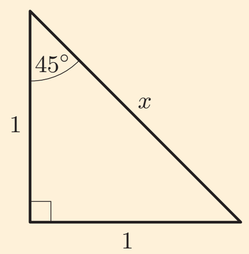
The right-angled triangle in Figure 17 can also be used to find the trigonometric ratios for 30°, the third angle in the triangle. You can see that the ratios are as follows:

$$\sin 30° = \frac{1}{2}, \quad \cos 30° = \frac{\sqrt{3}}{2} \quad \text{and} \quad \tan 30° = \frac{1}{\sqrt{3}}.$$

You can find the trigonometric ratios for the angle 45° by using a right-angled isosceles triangle in a similar way, in the next activity.

## Activity 11 *Finding the sine, cosine and tangent of 45°*

For the triangle below, find the length $x$ by using Pythagoras' Theorem, and then calculate $\sin 45°$, $\cos 45°$ and $\tan 45°$.




The trigonometric ratios for the angles 30°, 45° and 60° are used frequently, so they are listed in Table 1. If you study mathematics further, then you will find it useful to remember these values, or remember how to find them from the particular triangles discussed in this subsection.

**Table 1** Sine, cosine and tangent of special angles

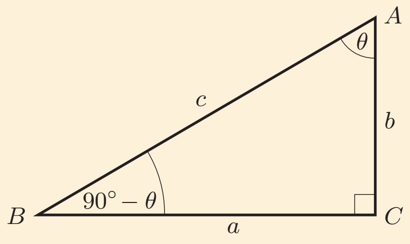
| $\theta$ | $\sin\theta$ | $\cos\theta$ | $\tan\theta$ |
|---|---|---|---|
| 30° | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{3}}$ |
| 45° | $\frac{1}{\sqrt{2}}$ | $\frac{1}{\sqrt{2}}$ | 1 |
| 60° | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $\sqrt{3}$ |

There are various relationships between sines, cosines and tangents of angles. These relationships can help you to remember the values in Table 1, and they are often helpful in other ways.

In the next activity you will find two such relationships.

## Activity 12 *Finding relationships between sines and cosines*

The diagram below shows a general right-angled triangle. Since its two acute angles add up to 90°, one is marked $\theta$ and the other is marked $90° - \theta$.




(a) Write down expressions for $\sin\theta$, $\cos\theta$, $\sin(90° - \theta)$ and $\cos(90° - \theta)$, in terms of the side lengths $a$, $b$ and $c$.

(b) Use the results of part (a) to show that if $\theta$ is an acute angle, then

$$\cos\theta = \sin(90° - \theta) \quad \text{and} \quad \sin\theta = \cos(90° - \theta).$$

In Activity 12 you were asked to prove the following two results.

$$\cos\theta = \sin(90^\circ - \theta)$$
$$\sin\theta = \cos(90^\circ - \theta)$$

These equations are examples of *identities*, as they are true for every acute angle $\theta$. They tell you that if two angles add up to 90°, then the sine of one angle is the cosine of the other, and vice-versa. For example,

$$\cos 30^\circ = \sin 60^\circ, \quad \cos 45^\circ = \sin 45^\circ \quad \text{and} \quad \cos 60^\circ = \sin 30^\circ.$$

This explains the repeated values that you can see in the sine and cosine columns of Table 1.

You can obtain another useful identity from the definitions of sine, cosine and tangent. The definitions are

$$\sin\theta = \frac{\text{opp}}{\text{hyp}}, \quad \cos\theta = \frac{\text{adj}}{\text{hyp}} \quad \text{and} \quad \tan\theta = \frac{\text{opp}}{\text{adj}}.$$

From these equations you can see that

$$\frac{\sin\theta}{\cos\theta} = \frac{\dfrac{\text{opp}}{\text{hyp}}}{\dfrac{\text{adj}}{\text{hyp}}} = \frac{\text{opp}}{\text{hyp}} \times \frac{\text{hyp}}{\text{adj}} = \frac{\text{opp}}{\text{adj}} = \tan\theta,$$

which gives the identity below.

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

You might like to check this identity for some of the values in Table 1.

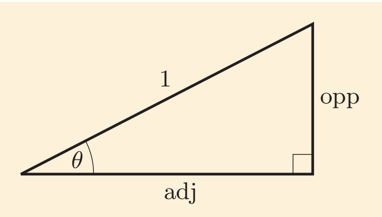



**Figure 18** A right-angled triangle with hypotenuse of length 1

Finally, a very neat identity can be obtained by considering a right-angled triangle whose hypotenuse has length 1, as shown in Figure 18, and using Pythagoras' Theorem. In this triangle,

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{\text{opp}}{1} = \text{opp} \quad \text{and} \quad \cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{\text{adj}}{1} = \text{adj}.$$

Thus, by Pythagoras' Theorem,

$$(\sin\theta)^2 + (\cos\theta)^2 = 1^2 = 1.$$

It is conventional to write $(\sin\theta)^2$ and $(\cos\theta)^2$ as $\sin^2\theta$ and $\cos^2\theta$, so the identity above is usually written as follows.

$$\sin^2\theta + \cos^2\theta = 1$$

Again, you might like to check this identity for some of the values in Table 1.

In Section 3 you will see that $\sin\theta$, $\cos\theta$ and $\tan\theta$ can also be defined for angles other than acute angles, and the identities above also hold for such angles.

# 2 Solving general triangles

There are many practical activities, such as surveying, navigating and designing buildings, where calculating unknown side lengths and angles in triangles is important. The process of finding some or all of the unknown side lengths or angles in a triangle is known as **solving the triangle**.

In Section 1 you saw how to solve a right-angled triangle. This section extends those ideas and explains how to solve triangles that do not necessarily have a right angle, and also how to find the area of a triangle when the height is not known. It considers only acute-angled triangles – an **acute-angled triangle** is one in which all three angles are acute. But you will see in Section 3 that the same techniques can also be used for triangles that have an obtuse angle.

Figure 19 The Franz Josef glacier

As an illustration of the kind of problem that can be solved by finding lengths and angles in triangles, you will see how to use the techniques in this section to estimate how fast a glacier is moving. This problem was investigated in the television series *Rough Science*. The scientists on the programme were challenged to estimate the speed at which the Franz Josef glacier in New Zealand (Figure 19) was moving, and you can see how they tackled the problem in the next activity.

### Activity 13 *Learning about the speed of a glacier*

Watch the excerpt from the television programme *Rough Science* to see how the scientists estimated the speed of the glacier.

Video

The video is on the DVD.

The *Rough Science* programmes were made by the BBC on behalf of The Open University.

In the video you saw that the scientists took the following steps to find the speed of the glacier.

- First they made a large protractor to measure angles to a precision of 0.1°, as shown in Figure 20.
- Then they placed a flag on the glacier, shown in Figure 21, and marked two points 50 m apart on the mountain along the edge of the glacier, near the flag.
- The angles from each point to the flag were measured on consecutive days with the large protractor. This pinpointed the position of the flag on each day.
- Finally, they used the angle measurements, a scale diagram and trigonometry to estimate how far the glacier had moved in a day.

Figure 20 Using the large protractor

In this section you will see how to solve the glacier problem, using two important trigonometric rules for finding unknown side lengths and angles in general triangles, known as the *Sine Rule* and the *Cosine Rule*.

You will learn about these rules in the next two subsections, and in the third subsection you will see how to apply them to find the speed of the glacier.

Figure 21 The flag on the glacier

## 2.1 The Sine Rule

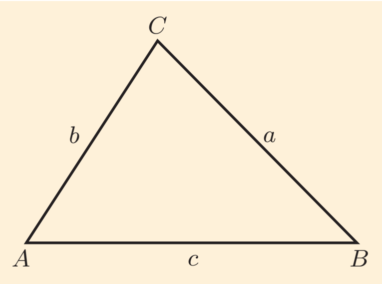



**Figure 22** How to label a general triangle

In this subsection and the next you will meet rules that relate the angles and side lengths of any triangle. These rules are most easily stated using the notation shown in Figure 22. Here the vertices of a triangle are labelled $A$, $B$ and $C$. The side lengths are labelled $a$, $b$ and $c$ in such a way that vertex $A$ is opposite side length $a$, vertex $B$ is opposite side length $b$, and vertex $C$ is opposite side length $c$. The angles at vertices $A$, $B$ and $C$ can be denoted by $\angle A$, $\angle B$ and $\angle C$, or just by $A$, $B$ and $C$.

The notation in Figure 22 is often used for a general triangle, as it helps you to remember which angle is related to which side, and it also makes the resulting formulas easier to remember. It will be used throughout this unit.

In Section 1 you saw how to solve right-angled triangles by using the trigonometric ratios sine, cosine and tangent, and in Unit 8 you saw that one way of making geometric problems easier to solve is to draw construction lines. So if you are trying to find a relationship between the sides and angles of the general triangle in Figure 22, then one approach is to draw a construction line to introduce right-angled triangles and use some of the facts that you already know about these.

Figure 23 shows a general triangle with three acute angles. A construction line has been drawn from $C$ at right angles to the opposite side. A line like this, drawn at right angles to another line, is called a **perpendicular** and the process of drawing such a line is called **dropping a perpendicular**.

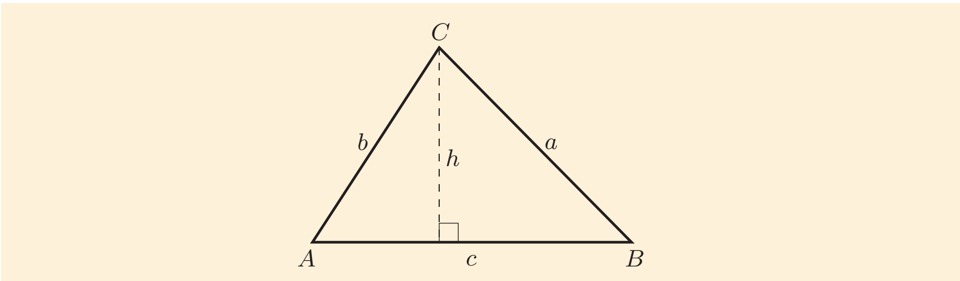



**Figure 23** A general triangle with a perpendicular added

Let $h$ be the length of the perpendicular. In the left-hand triangle you can use the fact that sine is opposite over hypotenuse to give

In this formula, $A$ is short for $\angle A$.

$$\sin A = \frac{h}{b}, \quad \text{so} \quad h = b \sin A.$$

In the right-hand triangle you can get a similar expression for the sine of the angle $B$:

$$\sin B = \frac{h}{a}, \quad \text{so} \quad h = a \sin B.$$

The two expressions above for $h$ are equal, so

$$b \sin A = a \sin B.$$

This equation can be rearranged to give the equation

$$\frac{a}{\sin A} = \frac{b}{\sin B}.$$

In a similar way, if you drop a perpendicular from the vertex $A$ to the opposite side of the triangle, then you can show that

$$\frac{b}{\sin B} = \frac{c}{\sin C}.$$

Combining this equation with the one above gives the following rule.

**Sine Rule**

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

or, equivalently,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

The second form of the Sine Rule is obtained from the first by taking reciprocals of each of the fractions.

The Sine Rule was first given a systematic treatment by the thirteenth-century Persian mathematician and polymath Nasir al-Din al-Tusi, who was also one of the first people to treat trigonometry as a separate subject rather than as part of astronomy.

The Sine Rule tells you that the ratio of a side length of a triangle to the sine of the angle opposite that side is the same no matter which side and its opposite angle you consider. So it can be used in the following way.

**Using the Sine Rule**

The Sine Rule can be used if you know one side length of a triangle and the opposite angle, and one further angle or side length.

For example, if you know the length $a$ and the opposite angle $A$, and also the angle $B$, then you can find the length $b$ by using the first equation in the Sine Rule,

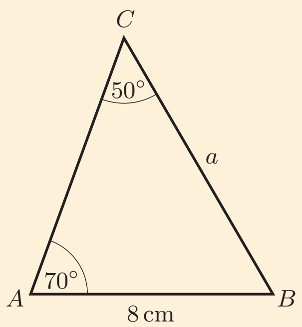
$$\frac{a}{\sin A} = \frac{b}{\sin B},$$

and rearranging this equation to make $b$ the subject.

The next two examples illustrate how this can be done.

**Example 6** *Using the Sine Rule to find a side length*

Find the length of the side $BC$ in the triangle below, to two significant figures.




**Solution**

A side length and the opposite angle are known, together with one further angle, so the Sine Rule can be used.

The side $AB$ of length 8 cm is opposite the $50°$ angle at $C$, so $c = 8$ and $C = 50°$. Also the side $BC$ of unknown length is opposite the $70°$ angle at $A$, so the unknown length is $a$ and $A = 70°$.

By the Sine Rule,

$$\frac{a}{\sin A} = \frac{c}{\sin C},$$

which gives

$$\frac{a}{\sin 70^\circ} = \frac{8}{\sin 50^\circ}, \quad \text{so} \quad a = \frac{8 \sin 70^\circ}{\sin 50^\circ} = 9.813\ldots.$$

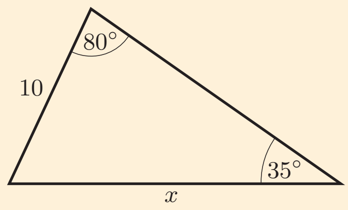
Hence the length $BC$ is 9.8 cm (to 2 s.f.).

In Example 6, the vertices and the sides were labelled as they are in the statement of the Sine Rule, and you may find it helpful to label your triangles in a similar way while you become familiar with the Sine Rule. When you feel confident using the Sine Rule, you can apply it directly without this labelling, as shown in the next example.

Tutorial clip

## Example 7 *Using the Sine Rule without vertex labels*

Find the length $x$ in the triangle below, to three significant figures.




### Solution

A side length and the opposite angle are known, so the Sine Rule can be used. The side length 10 is opposite the angle of 35°, and the unknown side length $x$ is opposite the angle of 80°.

By the Sine Rule,

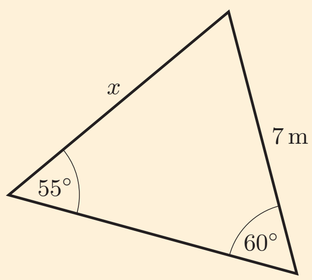
$$\frac{10}{\sin 35^\circ} = \frac{x}{\sin 80^\circ}, \quad \text{so} \quad x = \frac{10 \sin 80^\circ}{\sin 35^\circ} = 17.169\ldots.$$

Hence $x = 17.2$ (to 3 s.f.).

Here is a similar question for you to try.

## Activity 14 *Using the Sine Rule to find a side length*

Find the length $x$ in the triangle below, to three significant figures.



In this subsection you have seen that the Sine Rule is useful for solving a triangle when you know the length of a side and the angle opposite this side. As you have seen, if you also know one other angle, then you can use the Sine Rule to calculate the side length opposite this angle. If instead you know one further side length, then you can often use the Sine Rule to calculate the the angle opposite this side, but this is slightly more complicated and is covered in Section 3.

## 2.2 The Cosine Rule

Although the Sine Rule is useful in many cases, it can be used only if you know the length of a side and the opposite angle. If you know the lengths of two sides and the angle between them, as in Figure 24, then a different rule, called the *Cosine Rule*, can be used.

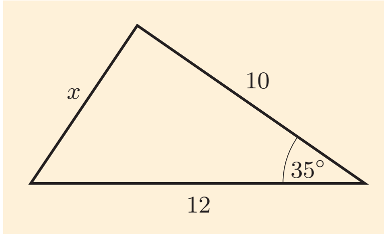



**Figure 24** A triangle with two known sides and a known angle between them

Figure 25 shows a general triangle with three acute angles. Suppose that you want to find a formula for the side length $a$ in terms of the side lengths $b$ and $c$, and the angle $A$. To do this, you can start by dropping a perpendicular from $C$, as shown. The side of length $c$ is then divided into two parts: if you call the length of one part $y$, then the other part has length $c - y$.

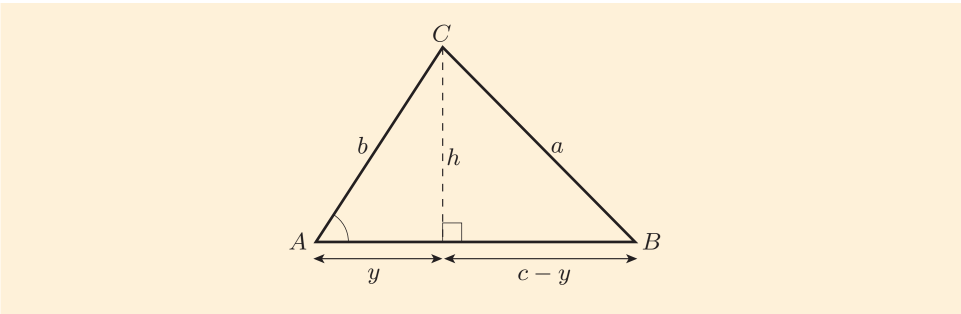



**Figure 25** A general triangle with a perpendicular added

Applying Pythagoras' Theorem to each of the right-angled triangles gives

$$b^2 = y^2 + h^2, \qquad (2)$$

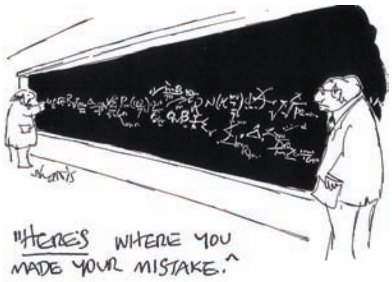
$$a^2 = (c - y)^2 + h^2.$$

Expanding the brackets in the second of these equations gives

$$a^2 = c^2 - 2cy + y^2 + h^2.$$

The right-hand side of this equation contains the expression $y^2 + h^2$, which is equal to $b^2$ by equation (2). So the equation above can be rearranged as

$$a^2 = b^2 + c^2 - 2cy. \qquad (3)$$




The length $y$ in this equation can be replaced by using the fact that the right-angled triangle on the left in Figure 25 gives

$$\cos A = \frac{y}{b}, \quad \text{so} \quad y = b \cos A.$$

Substituting this expression for $y$ in equation (3) gives the equation

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

This equation is one form of the Cosine Rule. There are three forms of this rule, one for each of the three angles, as shown overleaf.

Book II of Euclid's *Elements* contains a geometric theorem equivalent to the Cosine Rule. It was put into its present useful form by the fourteenth-century Persian mathematician Jamshid Al-Kashi, and is still known in France as 'Le Théorème d'Al-Kashi'.

### Cosine Rule

$$a^2 = b^2 + c^2 - 2bc\cos A,$$
$$b^2 = c^2 + a^2 - 2ca\cos B,$$
$$c^2 = a^2 + b^2 - 2ab\cos C.$$

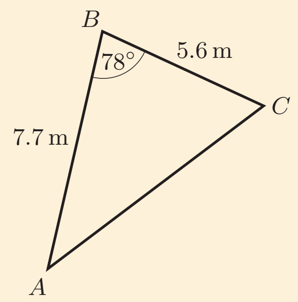
These three equations look quite complicated, but there are some patterns in the way that the letters appear that will help you to remember them. For example, only one length appears on the left-hand side of each equation, and this length is related to the angle whose cosine appears on the right-hand side. The other two lengths appear only on the right-hand side in each case.

You will see in the next section that if $\angle C$ is a right angle, then $\cos C = 0$. In this case, the third form of the Cosine Rule simply states that $c^2 = a^2 + b^2$. This is Pythagoras' Theorem, as you would expect in a right-angled triangle. In fact, the Cosine Rule is similar to Pythagoras' Theorem, with an extra correction term because the opposite angle is not a right angle in general.

The next two examples illustrate how to use the Cosine Rule.

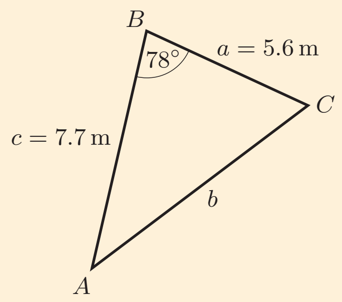
### Example 8 *Using the Cosine Rule to find a side length*

Use the Cosine Rule to find the length $AC$ in the triangle below, to two significant figures.




### Solution

Label the sides with letters and identify which form of the Cosine Rule to use. The known angle is angle $B$, so use the second form.



By the Cosine Rule,

$$b^2 = c^2 + a^2 - 2ca\cos B.$$

Substitute in the values and do the calculation.

Substituting $a = 5.6$, $c = 7.7$ and $B = 78°$ gives

$$b^2 = 7.7^2 + 5.6^2 - 2 \times 7.7 \times 5.6\cos 78° = 72.719\ldots.$$

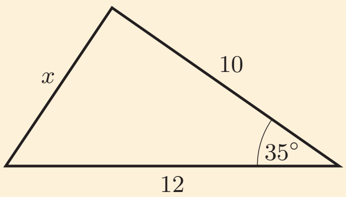
So $b = \sqrt{72.719\ldots} = 8.52\ldots$. Hence the length $AC$ is 8.5 m (to 2 s.f.).

As with the Sine Rule, once you are familiar with the Cosine Rule you may prefer to apply it without labelling the vertices and sides of the triangle.

Tutorial clip

## Example 9 *Using the Cosine Rule without vertex labels*

Find the length $x$ in the triangle below, to two significant figures.




### Solution

The lengths of two sides and the included angle are known, so use the Cosine Rule.

By the Cosine Rule,

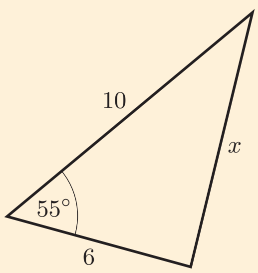
$$x^2 = 10^2 + 12^2 - 2 \times 10 \times 12\cos 35° = 47.403\ldots.$$

So $x = \sqrt{47.403\ldots} = 6.88\ldots = 6.9$ (to 2 s.f.).

Here is a similar question for you to try.

## Activity 15 *Using the Cosine Rule to find a side length*

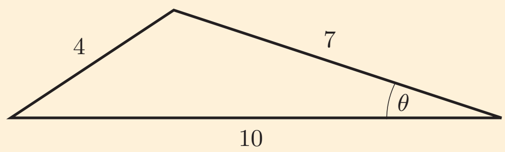
Find the length $x$ in the triangle below, to two significant figures.




The Cosine Rule can also be used if you know the lengths of all three sides of a triangle and you want to find an angle, as in the following example.

## Example 10 *Using the Cosine Rule to find an angle*

Find the angle $\theta$ in the triangle below, to the nearest degree.




### Solution

The lengths of three sides are known, so use the Cosine Rule.

By the Cosine Rule,

$$4^2 = 7^2 + 10^2 - 2 \times 7 \times 10 \cos\theta.$$

This equation simplifies to

$$16 = 149 - 140 \cos\theta.$$

Making $\cos\theta$ the subject of the equation gives:

$$140 \cos\theta = 133$$

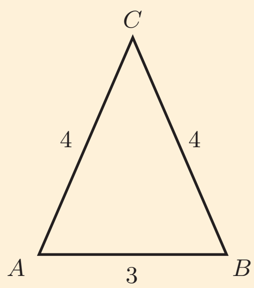
$$\cos\theta = \frac{133}{140} = 0.95.$$

So $\theta = \cos^{-1}(0.95) = 18.194\ldots^\circ = 18^\circ$ (to the nearest degree).

You can use a similar method in the following activity.

## Activity 16 *Finding all the angles in a triangle*

Use the Cosine Rule to calculate $\angle B$ of the isosceles triangle shown below, and deduce the other angles. Give your answers to the nearest degree.



## Summarising the methods of solving a triangle

In order to solve a triangle, you need to know two angles and one side length, or one angle and two side lengths, or three side lengths. You have now seen four methods of finding unknown lengths and angles in a triangle:

- Pythagoras' Theorem
- the trigonometric ratios sine, cosine and tangent
- the Sine Rule
- the Cosine Rule.

The process of choosing which of these methods to use can be summarised in the form of a decision tree, as shown in Figure 26.

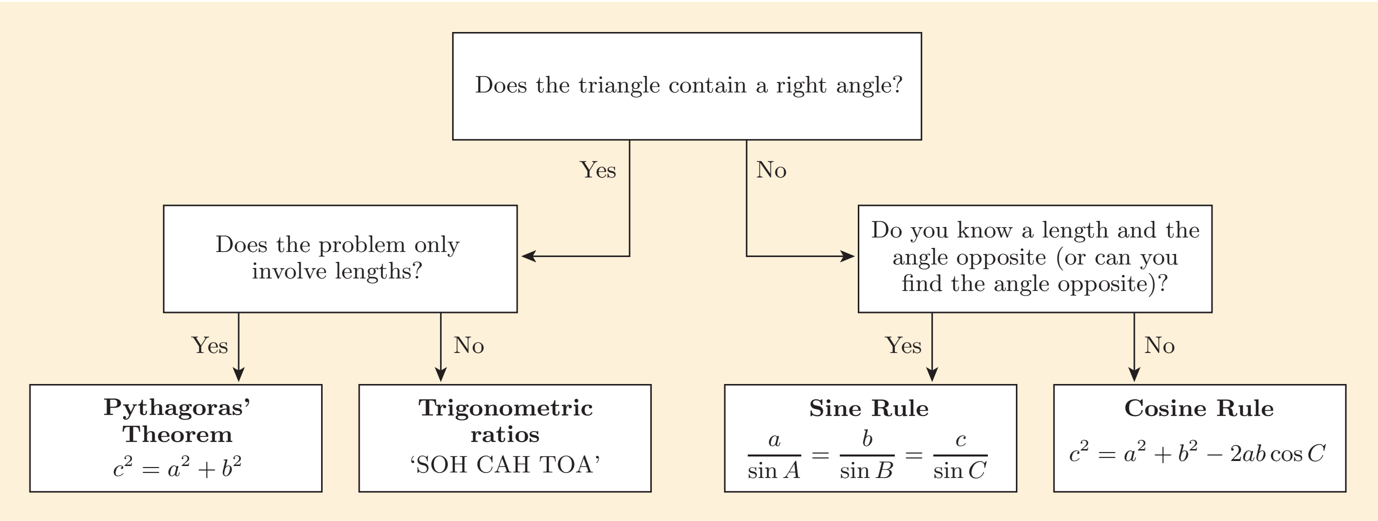
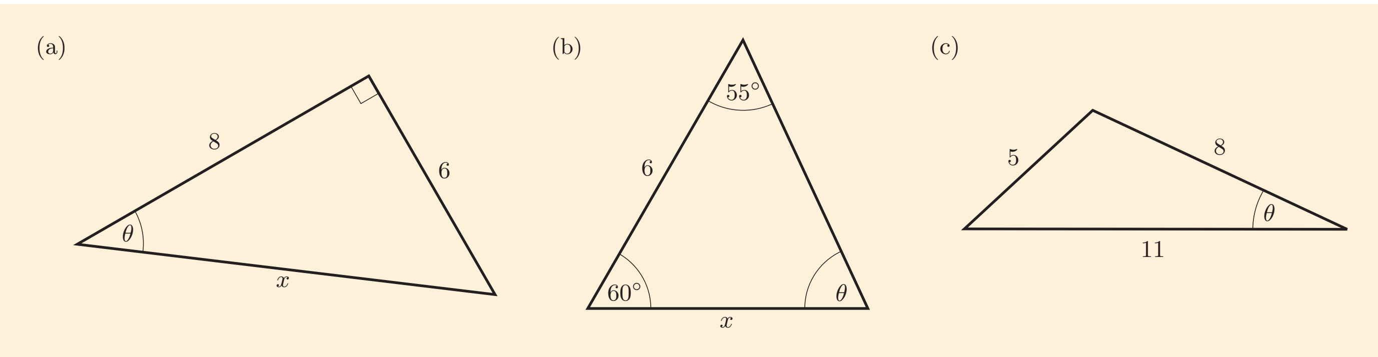



**Figure 26** A decision tree showing methods of solving a triangle

You can try using the decision tree in the following activity.

### Activity 17 *Choosing which way to solve a triangle*

For each of the following diagrams, decide which is the best method to use to find the unknown length $x$ and the unknown angle $\theta$, in either order. (You are not asked to find $x$ and $\theta$.)



## 2.3 Finding the speed of the glacier

Now that you have met the Sine Rule and the Cosine Rule, you can estimate the speed of the glacier in the *Rough Science* programme.

The method used in the programme was to place a flag on the glacier and make observations of the direction of the flag from each end of a 50 m line on the ground beside the glacier, which the scientists called their *baseline*. These observations were made at the same time on two consecutive days, and they then used trigonometry to estimate how far the flag moved in a day.

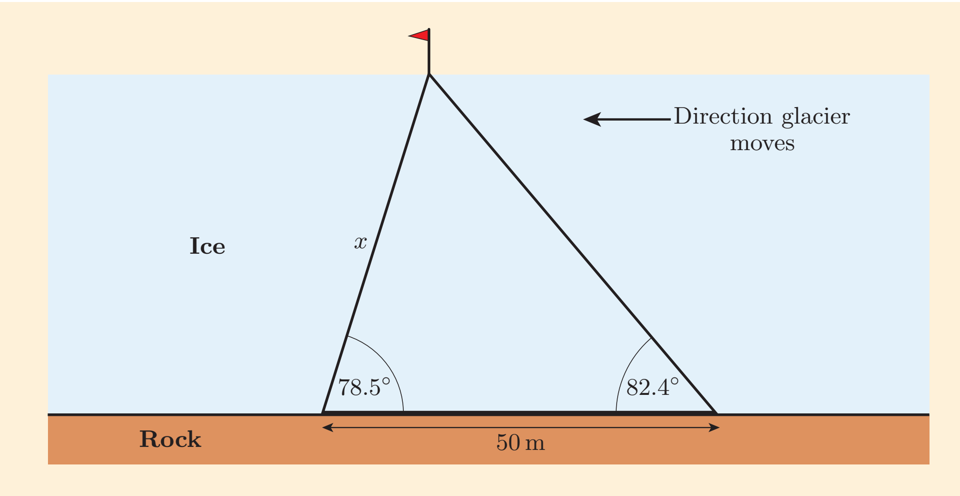
The scientists in the programme used a different trigonometric approach from the one here – there are often different ways to tackle a problem!

In the next activity you are asked to find the distance of the flag from one end of the baseline on each of the two days.

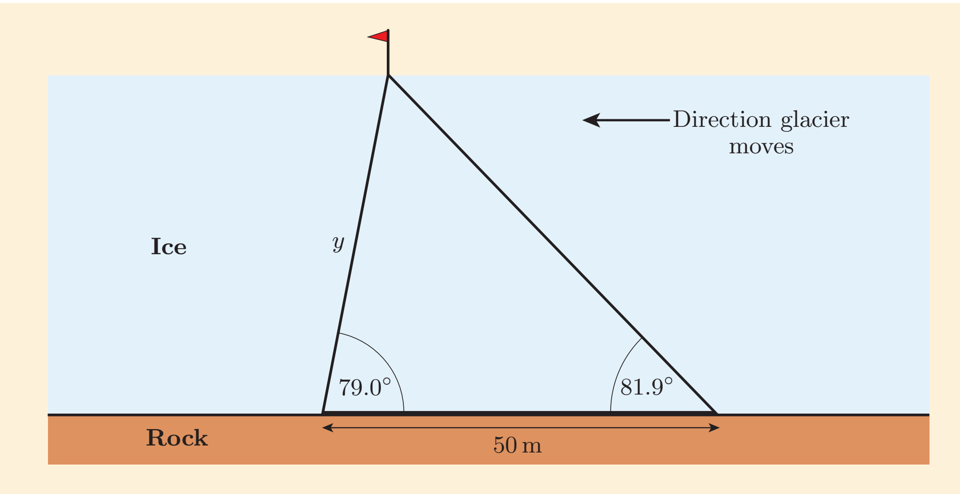
### Activity 18 *Finding the distance to the flag*

(a) On the first day the angles to the flag from each end of the 50 m baseline were 78.5° and 82.4°, as shown below. Calculate the length $x$ to the nearest centimetre.

In this and the next figure the angles have not been drawn accurately, in order to indicate that the triangle for the second day is not congruent to the one for the first day.




(b) On the second day the angles to the flag from each end of the 50 m baseline were 79.0° and 81.9°, as shown below. Calculate the length $y$ to the nearest centimetre.



By combining the results from the two parts of Activity 18, you obtain the diagram shown in Figure 27.

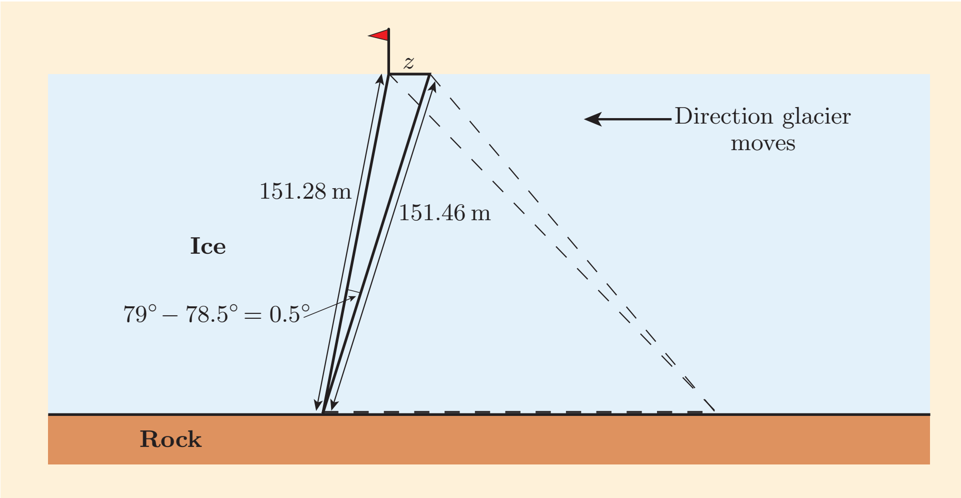



**Figure 27** A diagram showing the movement of the flag

The triangle with solid lines in Figure 27 has two known sides and a known angle between these sides. So you can use the Cosine Rule to find the distance $z$ that the flag moved.

**Activity 19** *Finding the movement of the flag*

Use the Cosine Rule in Figure 27 to find the distance $z$ that the flag moved in one day, to two significant figures.

In this activity you can use the rounded values of the side lengths and angles shown in Figure 27, since only a rough estimate of the speed of the glacier is required.

The answer found in Activity 19 agrees with the answer found in the video: the estimate of the speed of the glacier is 1.3 m per day.

## 2.4 A formula for the area of a triangle

In Unit 8, a formula was obtained for the area of a triangle in terms of the base and the height of the triangle. Using trigonometry you can derive a formula for the area of a triangle that can be used if you know the lengths of two sides and the included angle. As with the Sine Rule and Cosine Rule, we will derive this formula for a triangle with three acute angles, but it can also be used for triangles with an obtuse angle.

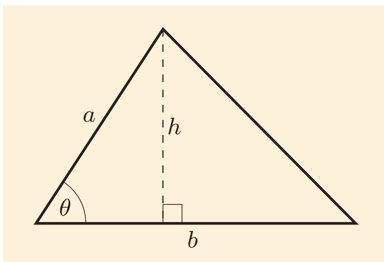



**Figure 28** Finding the area of a triangle

For the triangle in Figure 28, the base is $b$ and the height is $h$. So the area of the triangle is $\frac{1}{2}bh$, as given in Unit 8, Section 4. From the right-angled triangle on the left,

$$\sin\theta = \frac{h}{a}, \quad \text{so} \quad h = a\sin\theta.$$

Substituting this formula for $h$ into $\frac{1}{2}bh$ gives the area of the triangle as

$$\tfrac{1}{2}bh = \tfrac{1}{2}b \times a\sin\theta = \tfrac{1}{2}ab\sin\theta.$$

This gives the following formula for the area of a triangle.

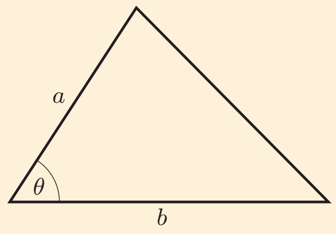



**Figure 29** A triangle with two sides of lengths $a$ and $b$ and included angle $\theta$

***Area of a triangle***

The area of a triangle with two sides of lengths $a$ and $b$, and included angle $\theta$, is

$$\text{area} = \tfrac{1}{2}ab\sin\theta.$$

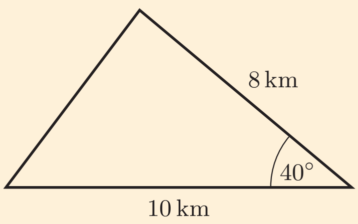
This area formula is more useful in practical situations than the formula $\frac{1}{2}bh$. For example, in surveying it gives the area of a triangle of land in terms of quantities that a surveyor can readily measure. The areas of most pieces of land can be found by breaking them down into convenient triangular shapes and using this formula in each triangle.

You can use the formula in the following activity.

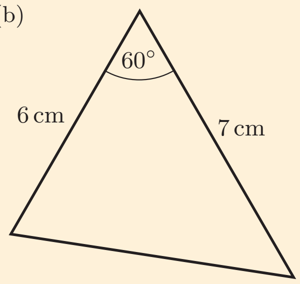
**Activity 20** *Finding the areas of triangles*

Find the area of each of the following triangles, correct to three significant figures.

(a)




(b)




In many cases calculating the area of a triangle would be easier if there were a simple formula for the area of a triangle in terms of just the lengths of the three sides. Such a formula was derived by the Greek mathematician Heron around AD 62 and later found independently by Chinese mathematicians.

Heron of Alexandria taught physics and engineering, as well as mathematics. Heron's Formula is derived in Book 1 of his textbook *Metrica*.

***Heron's Formula***

The area of a triangle with sides of length $a$, $b$ and $c$ is

$$\text{area} = \sqrt{s(s-a)(s-b)(s-c)},$$

where $s = \frac{1}{2}(a+b+c)$.

Note that the lengths $a$, $b$ and $c$ appear symmetrically in the formula – this makes it more memorable.

The quantity $s$ in Heron's Formula is called the **semi-perimeter** of the triangle because it is half the perimeter.

Heron's Formula can be proved by expressing one angle of the triangle in terms of the lengths of the sides and then using the formula $\frac{1}{2}ab\sin\theta$ for the area of the triangle. However, it takes considerable algebraic skill to write the final result in the neat form given in Heron's Formula.

In the next activity you are asked to use the two area formulas from this subsection to calculate the area of an equilateral triangle.

**Activity 21** *Using area formulas*

For an equilateral triangle of side length 2:

(a) Calculate the area of the triangle using the formula: area $= \frac{1}{2}ab\sin\theta$.

(b) Calculate the semi-perimeter $s$ of the triangle and then use Heron's Formula to calculate the area of the triangle.

Give your answers in surd form.

# 3 Trigonometric functions

So far in this unit the trigonometric ratios sine, cosine and tangent have always been applied to angles between 0° and 90°. However, if you use your calculator to find $\sin 150°$, $\cos 1000°$ or even $\tan(-45°)$, then it gives the answers

$$\sin 150° = \tfrac{1}{2}, \quad \cos 1000° = 0.173\ldots, \quad \tan(-45°) = -1.$$

What do these values mean, and why are they needed?

In this section you will learn how to define the sine and cosine of every possible angle, and the tangents of most angles, and you will also see why defining these values is useful.

One reason why it is useful to define the sine and cosine of an angle greater than 90° is that some triangles have an obtuse angle, that is, an angle between 90° and 180°. In order to solve such triangles, you may want to use the Sine Rule or the Cosine Rule: these apply to triangles with an obtuse angle in the same way as they apply to other triangles. But to do that you need to know what the sine and cosine of an obtuse angle are!

Since the definitions of sine and cosine given in Section 1 do not make sense for an obtuse angle, new definitions are needed. But once you know these new definitions, it is natural to define the sine and cosine of *any* angle, not just acute and obtuse ones.

The sines and cosines of general angles give rise to functions, called *trigonometric functions*, that turn out to be useful in situations that are not explicitly related to triangles. For example, these functions can be used to model many types of real-world behaviour with a repeating nature, such as the occurrence of high tides – as you will see.

## 3.1 Sine, cosine and tangent of a general angle

Unit 8, Section 1, introduced the idea that an angle is a measure of rotation, or amount of turning, that can be measured in degrees. There, angles were discussed that take values up to and including 360°, that is, up to a rotation through one full turn. But it is possible to have a rotation through more than one full turn, so it makes sense to use general angles that measure more than 360°. You can also have rotations that are clockwise or anticlockwise, and it is useful to distinguish between these.

***Sign of an angle***

A general angle is a measure of rotation around a point, measured in degrees. Positive angles give anticlockwise rotations, and negative angles give clockwise rotations.

Some examples of angles corresponding to rotations around the origin, from the positive $x$-axis, are shown in Figure 30. The arrow indicates whether the rotation is anticlockwise or clockwise.

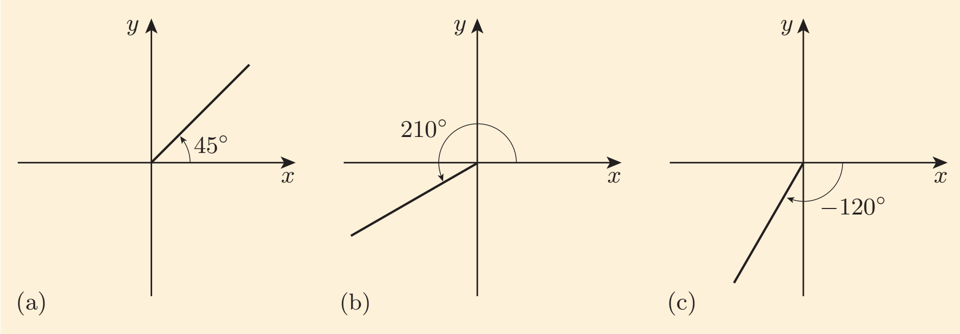



**Figure 30** Some general angles

The sine and cosine of a general angle $\theta$ are defined by using a point $P$ on the circle with radius 1 centred on the origin, which is called the **unit circle**. The position of the point $P$ is determined by the angle $\theta$; it is obtained by a rotation around the origin through the angle $\theta$ starting from the point on the $x$-axis with $x$-coordinate 1. If the angle $\theta$ is positive, then the rotation is anticlockwise; if the angle is negative, then the rotation is clockwise.

Some examples of how general angles give rise to points on the unit circle are shown in Figure 31.

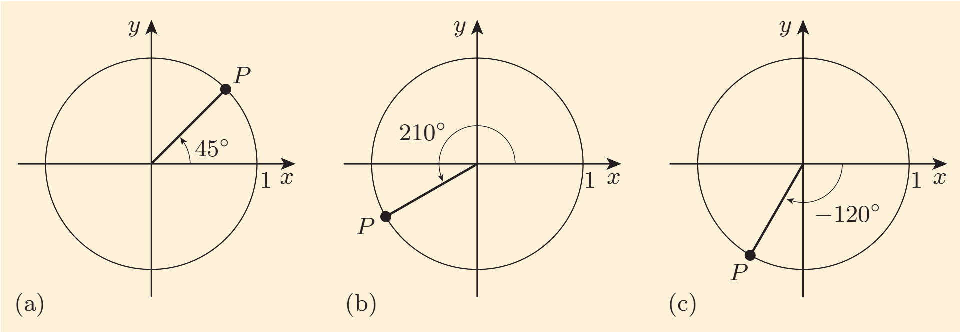



**Figure 31** General angles and points on the unit circle

In Figure 31, the $x$- and $y$-axes divide the graph into four regions, each of which is known as a **quadrant**. These quadrants are numbered in order anticlockwise as shown in Figure 32, and they are useful for describing where points, such as those on the unit circle, lie.

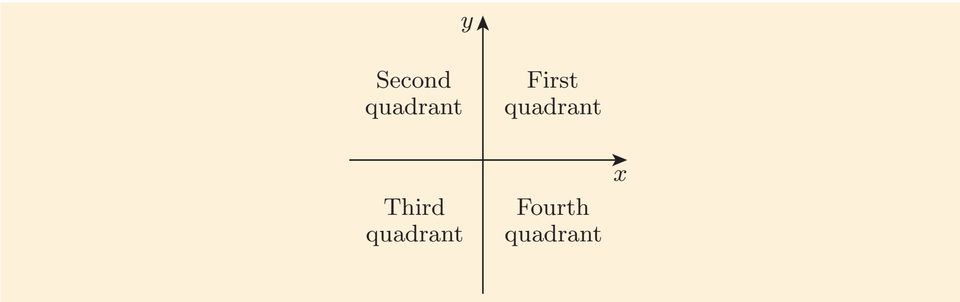



**Figure 32** The four quadrants

## Activity 22 *Plotting general angles on the unit circle*

For each of the following angles, draw a sketch to illustrate how the point $P$ on the unit circle is rotated through that angle from its starting position on the $x$-axis. Your sketches should be similar to those in Figure 31. In each case state the quadrant in which $P$ lies.

(a) $60°$ (b) $225°$ (c) $390°$ (d) $-70°$

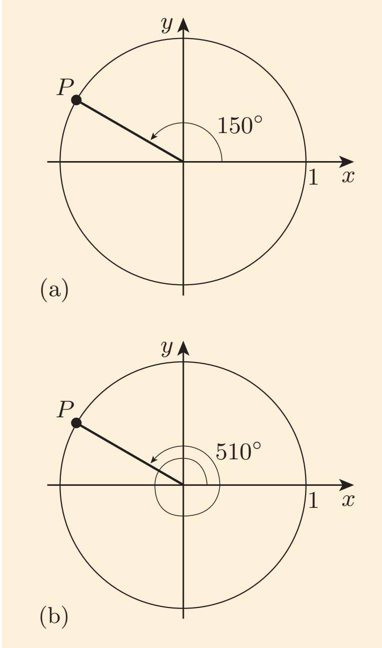



**Figure 33** Angles of 150° and 510°

It is possible that two different angles of rotation lead to the point $P$ being in the same position. For example, as you can see in Figure 33, the point $P$ is in the same position on the unit circle for the angle 510° as for the angle 150°. This is because

$$510° = 150° + 360°,$$

so 510° gives exactly one full turn around the origin more than 150°.

Figure 34 shows the point $P$ after it has rotated through an acute angle $\theta$, so $P$ is in the first quadrant.

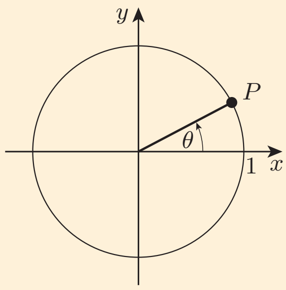



**Figure 34** The point $P$ after a rotation through an acute angle

If you drop a perpendicular from $P$ to the $x$-axis, then you obtain a right-angled triangle with hypotenuse of length 1 in which one angle is $\theta$, as shown in Figure 35.

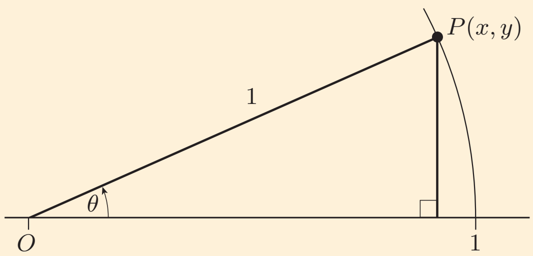



**Figure 35** A right-angled triangle with angle $\theta$

You can see that if $P$ has coordinates $(x, y)$, then

$$\sin\theta = \frac{y}{1} \quad \text{and} \quad \cos\theta = \frac{x}{1},$$

which give

$$x = \cos\theta \quad \text{and} \quad y = \sin\theta.$$

These two equations are the key to defining the sine and cosine of a general angle. This is done as follows.

### *Sine and cosine of a general angle*

For a general angle $\theta$, let $P$ be the point on the unit circle obtained by a rotation of $\theta$ around the origin from the positive $x$-axis, and suppose that $P$ has coordinates $(x, y)$. Then

$$\cos\theta = x \quad \text{and} \quad \sin\theta = y.$$

So the cosine and sine of a general angle $\theta$ are just the $x$- and $y$-coordinates of the point $P$ whose position on the unit circle is determined by the angle $\theta$, as shown in Figure 36.

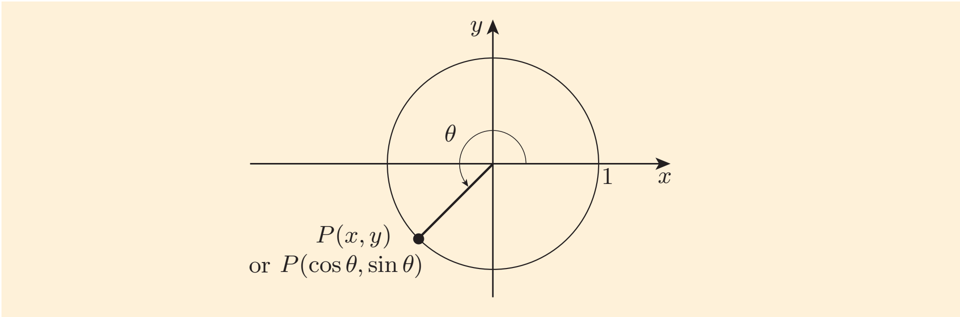



**Figure 36** Defining $\cos\theta$ and $\sin\theta$

To illustrate this definition, consider the angle $\theta = 150°$. For this angle, the point $P$ lies in the second quadrant, as shown in Figure 37.

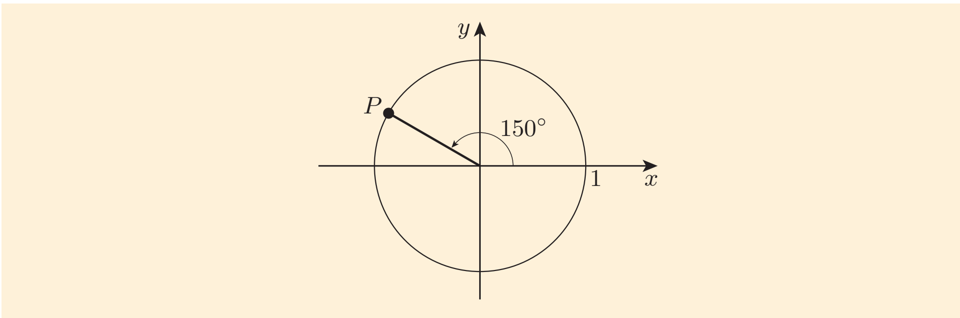



**Figure 37** The point $P$ obtained from the angle 150°

To find $\sin\theta$ and $\cos\theta$ in this case, you need to find the coordinates of this point $P$. You can do this by using the right-angled triangle shown in Figure 38, which has one angle equal to $180° - 150° = 30°$.

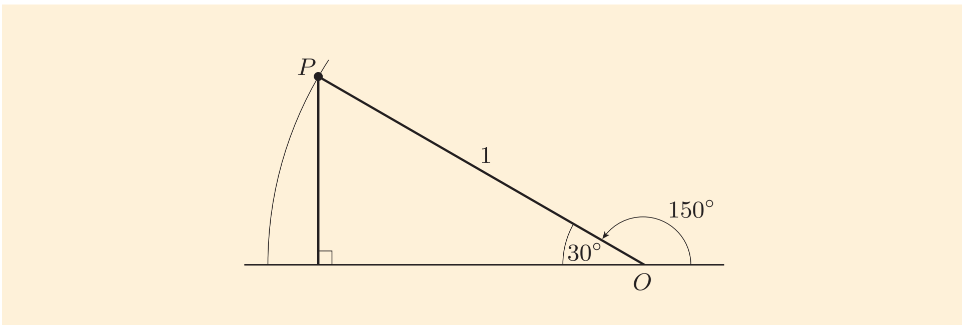



**Figure 38** A right-angled triangle related to the angle 150°

In the triangle in Figure 38, the hypotenuse is of length 1, so

$$\sin 30° = \frac{\text{opp}}{1} \quad \text{and} \quad \cos 30° = \frac{\text{adj}}{1}.$$

Hence

- the side opposite the angle of 30° has length $\sin 30° = \frac{1}{2}$,
- the side adjacent to the angle of 30° has length $\cos 30° = \frac{\sqrt{3}}{2}$.

You can find the values of sin 30° and cos 30° either by using your calculator or from Table 1 on page 73.

Therefore the coordinates of the point $P$ are $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

So, by the definitions of the sine and cosine of a general angle,

$$\cos 150° = -\frac{\sqrt{3}}{2} = -0.866\ldots \quad \text{and} \quad \sin 150° = \frac{1}{2} = 0.5.$$

You can check that your calculator gives these values for cos 150° and sin 150°.

Note that the values of cos 150° and sin 150° were found by using the values of cos 30° and sin 30°, respectively.

It is straightforward to write down the sine or cosine of an angle that is a multiple of 90°, because for such an angle the point $P$ lies on one of the axes. For example, Figure 39 shows that $\cos 0° = 1$ and $\sin 0° = 0$.

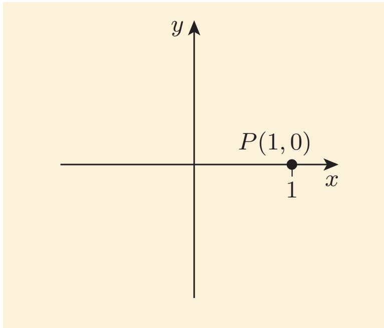



**Figure 39** $P$ is at $(1, 0)$ when $\theta = 0°$

Similarly, Figure 40 shows that $\cos 90° = 0$ and $\sin 90° = 1$.

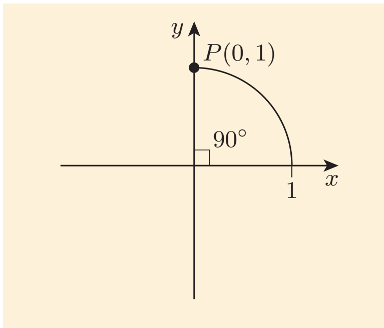



**Figure 40** $P$ is at $(0, 1)$ when $\theta = 90°$

Usually, you should simply use your calculator to find the sine and cosine of a general angle, just as you would do for an acute angle. But to make sure that you understand these definitions, it is worth working out some sines and cosines from basic principles. You are asked to do this in the following activity.

## Activity 23 *Finding sines and cosines from the definition*

(a) Find cos 225° and sin 225° (to four decimal places) by plotting the appropriate point $P$ on the unit circle and using a suitable right-angled triangle. Check your answers with a calculator.

(b) Find cos(−180°) and sin(−180°) by plotting the appropriate point $P$ on the unit circle. Check your answers with a calculator.

Now that you know the definition of the sine and cosine of any angle, the next question is: How can you define the tangent of a general angle? To answer this question, recall the relationship between the sine, cosine and tangent of an acute angle found in Subsection 1.4. In any right-angled triangle with an acute angle $\theta$,

$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

This equation can be used to *define* the tangent of a general angle.

### *Tangent of a general angle*

For a general angle $\theta$,

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \text{provided that } \cos\theta \neq 0.$$

Remember that division by 0 is not allowed.

The sines and cosines of 150° and 90° were found just before Activity 23.

For example,

$$\tan 150° = \frac{\sin 150°}{\cos 150°} = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}} = -0.577\ldots.$$

You can check this value on a calculator.

However, $\tan 90°$ cannot be defined because $\cos 90° = 0$.

**Activity 24** *Finding tangents from the definition*

Use your answers to Activity 23 to find $\tan 225°$ and $\tan(-180°)$.

Earlier in the subsection, the sine and cosine of a general angle $\theta$ were defined geometrically in terms of the position of a point $P$ on the unit circle. There is also a geometric interpretation of the tangent of $\theta$. This interpretation will be useful in the next subsection when we draw graphs of the sine, cosine and tangent functions.

First consider the situation when the angle $\theta$ is acute, so the point $P$ is in the first quadrant, as shown in Figure 41. A vertical line has been drawn through the point on the $x$-axis with coordinate 1, and the line from the origin through $P$ has been extended to meet this vertical line, at the point $Q$. A new right-angled triangle $\triangle OQN$ has thus been formed in which the side adjacent to the angle $\theta$ has length 1. In this triangle,

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{\text{opp}}{1} = \text{opp},$$

so the $y$-coordinate of $Q$ is $\tan\theta$.

The name tangent arises from the fact that in this diagram the line that passes through the point $(1, 0)$ and the point $Q$ is a *tangent* to the unit circle; that is, it meets this circle at exactly one point.

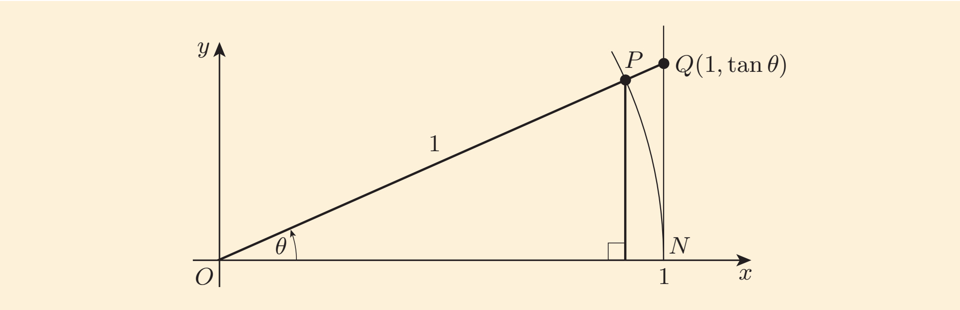



**Figure 41** The geometric interpretation of $\tan\theta$ when $\theta$ is acute

As the angle $\theta$ increases, the point $Q$ moves up the vertical line, so $\tan\theta$ increases. As $\theta$ reaches the value 90°, the line from the origin through $P$ becomes vertical. At this point $\tan 90°$ is not defined.

For values of $\theta$ between 90° and 180°, the line through the origin and $P$ once again meets the vertical line on the right, though now the point of intersection is *below* the $x$-axis, as shown in Figure 42. It can be shown, by considering the triangles in the diagram, that the $y$-coordinate of $Q$ is again $\tan\theta$, though the details are not given here.

It can also be shown that this interpretation works for any angle $\theta$ where $\tan\theta$ is defined; that is, the value of $\tan\theta$ is always the $y$-coordinate of the point $Q$. You will see an animation illustrating this fact in the next subsection.

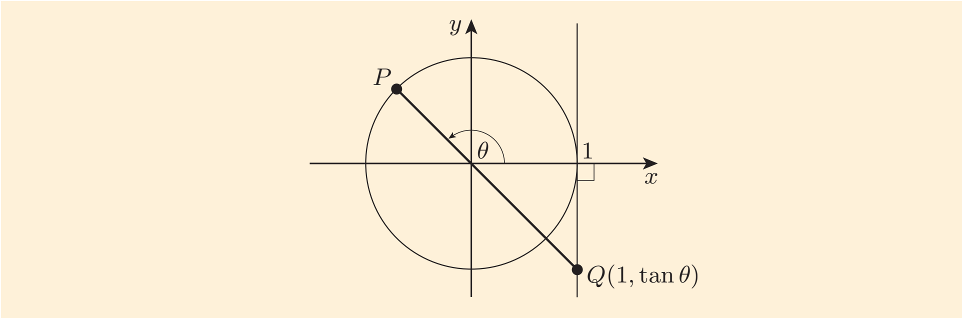



**Figure 42** The geometric interpretation of $\tan\theta$ when $\theta$ is obtuse

## 3.2 Graphs of sine, cosine and tangent

In Subsection 3.1 you saw how the sine, cosine and tangent of any angle are defined – except when division by zero is involved. So you can think of each of the expressions

$$\sin\theta, \quad \cos\theta \quad \text{and} \quad \tan\theta$$

as a 'rule' that takes an input value $\theta$ and produces an output value. In Unit 6, Section 3, you saw that a rule of this type, which transforms an input value into an output value, is often called a 'function'. For this reason, sine, cosine and tangent are often called **trigonometric functions**.

To gain a better understanding of a function, it is often helpful to plot its graph, as you saw with quadratics in Unit 10. Let's begin by plotting the graphs of sine, cosine and tangent in the range 0° to 90°. This can be done by using the values in Table 2, which were obtained from a calculator and are stated to two decimal places.

**Table 2** Values of trigonometric functions

| $\theta$ | 0° | 10° | 20° | 30° | 45° | 60° | 70° | 80° | 90° |
|---|---|---|---|---|---|---|---|---|---|
| $\sin\theta$ | 0.00 | 0.17 | 0.34 | 0.50 | 0.71 | 0.87 | 0.94 | 0.98 | 1.00 |
| $\cos\theta$ | 1.00 | 0.98 | 0.94 | 0.87 | 0.71 | 0.50 | 0.34 | 0.17 | 0.00 |
| $\tan\theta$ | 0.00 | 0.18 | 0.36 | 0.58 | 1.00 | 1.73 | 2.75 | 5.67 | – |

Using these values, the following graphs can be drawn. Only a few significant points are shown on the graphs.

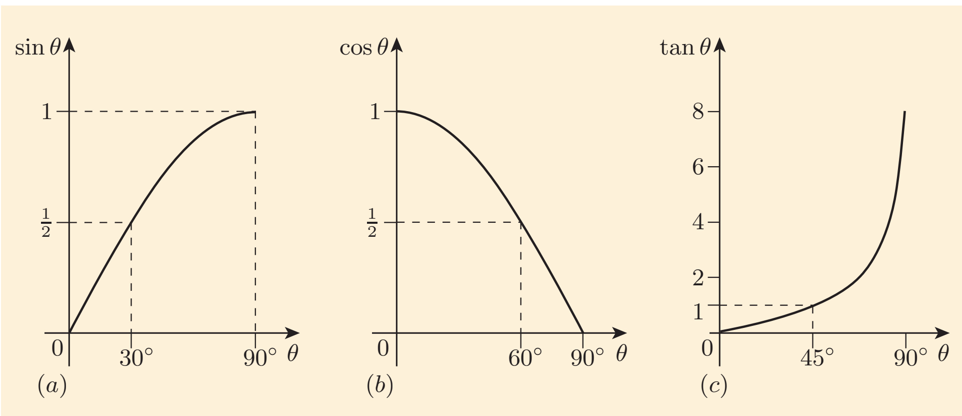



**Figure 43** Graphs of sine, cosine and tangent functions for $0° < \theta < 90°$

In Unit 14 the variable $y$ *will* be used on the vertical axis for such trigonometric graphs.

In Figure 43, the input variable $\theta$ is on the horizontal axis. The vertical axes are labelled with the names of the functions, in order to emphasise which function is involved and also to avoid using the variable $y$, which had a different meaning earlier in this section as one of the coordinates of the point $P$.

You can see that, as the angle $\theta$ increases from 0 to 90°:

- the value of $\sin\theta$ increases from 0 to 1;
- the value of $\cos\theta$ decreases from 1 to 0;
- the value of $\tan\theta$ increases from 0 and takes values that are arbitrarily large as the angle $\theta$ approaches 90°.

These changes take place because, as the point $P$ moves anticlockwise round the part of the unit circle in the first quadrant (see Figure 34 on page 89), the vertical coordinate of $P$, which is equal to $\sin\theta$, increases from 0 to 1, and the horizontal coordinate of $P$, which is equal to $\cos\theta$, decreases from 1 to 0. Also the point $Q$ (see Figure 41 on page 92) moves upwards from the point $(1, 0)$, arbitrarily far as $\theta$ approaches 90°.

But what do these graphs look like for other values of $\theta$? In fact, the other parts of these graphs have similar shapes to the parts that are plotted in Figure 43, but these other parts are positioned differently in relation to the axes. The following activity allows you to see how these graphs are generated as the point $P$ moves on the unit circle in the way described in the previous subsection.

Dynamic trigonometry

**Activity 25** *Using animated graphs of trigonometric functions*

Carry out the computer activity to explore the graphs of sine, cosine and tangent for different ranges of angles.

In Activity 25 you explored the graphs of the sine, cosine and tangent functions between −360° and 360°. In fact, the graphs repeat endlessly along the horizontal axis. The parts of the graphs corresponding to values of $\theta$ between −360° and 720° are shown in Figures 44, 45 and 46.

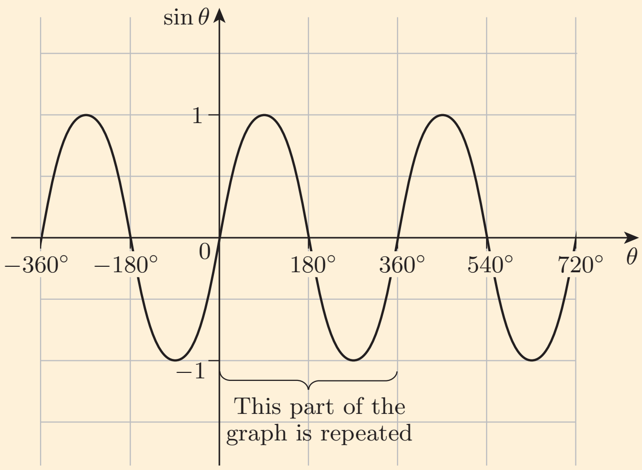



**Figure 44** The graph of the sine function

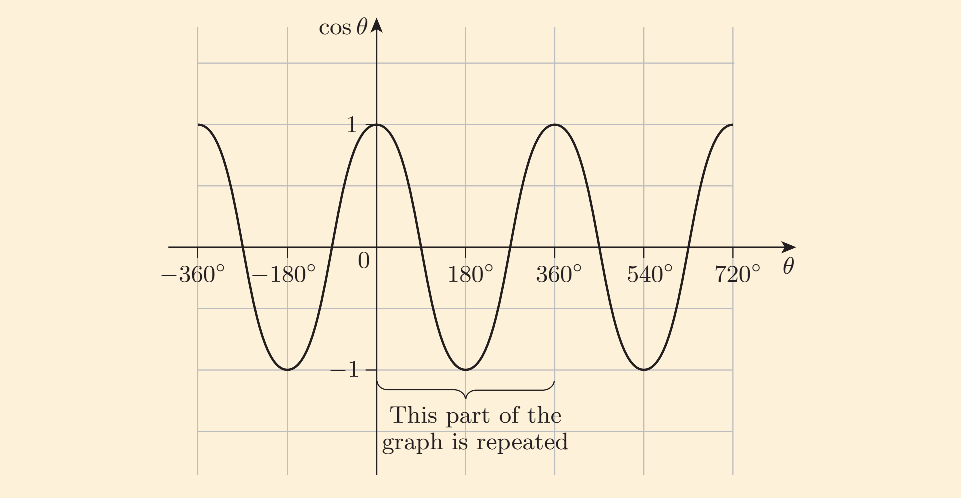



**Figure 45** The graph of the cosine function

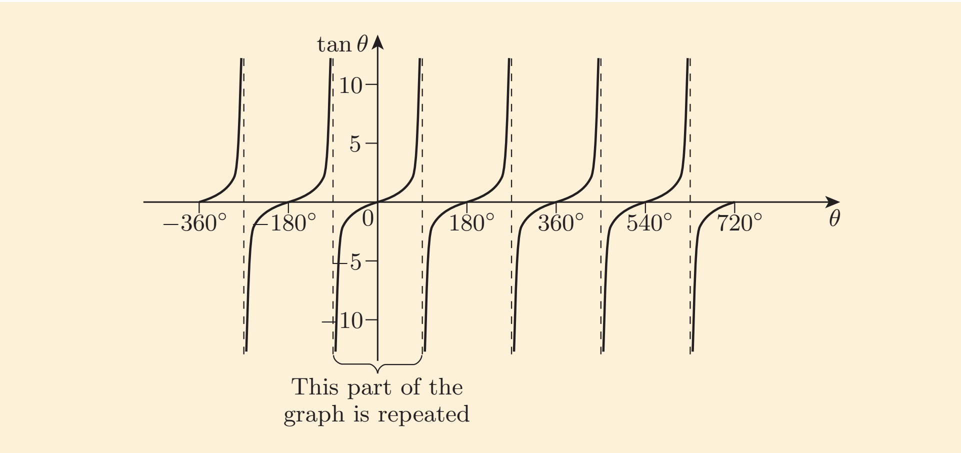



**Figure 46** The graph of the tangent function

The graphs of these three trigonometric functions have various properties, such as symmetry characteristics, which can be useful when you are working with sines, cosines and tangents. Let's have a look at some of these properties.

## Periodicity

A key feature of these graphs is that their shapes repeat in a regular way. The sine and cosine graphs repeat every $360°$ and we say that these functions are *periodic*, with **period** $360°$. This is what you would expect because as the angle $\theta$ increases the position of the point $P$ on the unit circle repeats every $360°$. The tangent function is also periodic but with a smaller period, of $180°$. So the values of the tangent function repeat twice as often as the values of the sine and cosine functions.

Because of this property of periodicity, an equation such as $\sin\theta = 0.5$ has infinitely many solutions, and not just the solution $\sin^{-1}(0.5) = 30°$. For example, $\sin 390° = \sin 30° = 0.5$ because $390° = 360° + 30°$. Another solution is $150°$, since $\sin 150° = 0.5$ as you saw in Subsection 3.1.

In Unit 14 you will see how to find all the solutions of equations such as $\sin\theta = 0.5$.

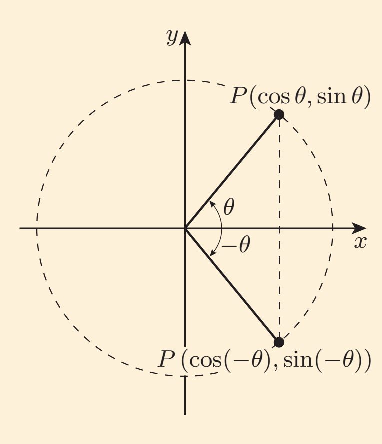



**Figure 47** $\cos(-\theta) = \cos(\theta)$

## Mirror symmetry

The sine and cosine graphs have *mirror symmetry* in any vertical line through a peak or trough on the graph. For example, the graph of the cosine function (part of which is shown in Figure 45) has mirror symmetry in the vertical axis. This property means that any angle and its negative have the same cosine value. This is what you would expect, because if the point $P$ is rotated around the origin from the positive $x$-axis by an angle of either $\theta$ or $-\theta$, as in Figure 47, then the resulting $x$-coordinate is the same. Facts like this can be written down as trigonometric identities; this fact gives

$$\cos(-\theta) = \cos(\theta)$$

for any angle $\theta$. For example, $\cos(-30°) = \cos 30°$.

## Rotational symmetry

Each of the three graphs has *rotational symmetry* about any point where the graph crosses the $\theta$-axis – if you rotate the graph through a half-turn about such a point, then it lies exactly on top of where it was before. For example, the sine graph (part of which is shown in Figure 44) has rotational symmetry about the origin. This means that any angle $\theta$ and its negative $-\theta$ have sine values that have the same magnitude, but one of the sine values is negative while the other is positive. Exactly the same is true of the tangent graph; these two facts give the identities

Remember that the *magnitude* of a number is its value without its negative sign, if it has one.

$$\sin(-\theta) = -\sin(\theta) \quad \text{and} \quad \tan(-\theta) = -\tan(\theta)$$

for any angle $\theta$. For example, $\sin(-30°) = -\sin 30°$.

## Asymptotes

The graph of the tangent function differs from the other two graphs in that it is broken up into separate pieces and it takes values that are arbitrarily large. The breaks in the graph correspond to the values of $\theta$ where $\cos\theta$ is zero, such as $\theta = 90°$.

For angles just below 90°, you can see that $\tan\theta$ is very large and positive. This is because $\tan\theta = \sin\theta/\cos\theta$, and $\sin\theta$ is close to 1 whereas $\cos\theta$ is very small and positive (if you divide a number close to 1 by a very small number, then the answer is a very large number). Similarly, for angles just above 90°, you can see that $\tan\theta$ is very large and negative.

This behaviour is described by saying that the tangent graph has an **asymptote** at $\theta = 90°$. Informally, an asymptote is a line that a graph approaches but never reaches. Asymptotes are often indicated by dashed lines, as in Figure 46.

**Activity 26** *The asymptotes of the tangent graph*

Write down the values of $\theta$ between 360° and 720° at which the asymptotes of the tangent graph occur.

## Relationships between the sine and cosine graphs

Another key fact that you saw in Activity 25 is that the sine and cosine graphs have the same basic shape. Various relationships between sines and cosines can be obtained from this fact.

For example, try the following 'thought experiment'. Imagine walking from left to right along the sine graph, starting from the origin. As the angle increases from 0°, the height you are above the horizontal axis increases from 0 to 1, then decreases from 1 through 0 to −1, and then increases from −1 to 0 again, and so on.

If you performed the same thought experiment with the cosine graph, again walking from left to right but this time starting at −90°, then the heights would follow exactly the same pattern. This is because the graph of the cosine function is obtained by shifting the graph of the sine function to the left by a distance of 90° on the $\theta$-axis. The corresponding trigonometric identity is

$$\sin\theta = \cos(-90^\circ + \theta)$$

for any angle $\theta$. For example, $\sin 60^\circ = \cos(-90^\circ + 60^\circ) = \cos(-30^\circ)$.

Similarly, if you walked along the cosine graph starting from 90°, but this time from right to left, then the same pattern of heights would occur. In this case, the resulting trigonometric identity is

$$\sin\theta = \cos(90^\circ - \theta) \tag{4}$$

Another way to see identity 4 is to plot

$$y = \sin\theta \quad\text{and}\quad y = \cos(90 - \theta)$$

on Graphplotter. One graph will lie on top of the other, illustrating that

$$\sin\theta = \cos(90^\circ - \theta)$$

for all the values of $\theta$ plotted on the graph.

for any angle $\theta$. For example, $\sin 60^\circ = \cos(90^\circ - 60^\circ) = \cos 30^\circ$.

You have already seen that the identity $\sin\theta = \cos(90^\circ - \theta)$ holds for acute angles, in Subsection 1.4. So now you know that it holds for *all* angles.

Similarly, the identity

$$\cos\theta = \sin(90^\circ - \theta)$$

for any angle $\theta$, can be seen from the sine and cosine graphs by using a thought experiment similar to those above; you might like to try this.

## Two other trigonometric identities

In Subsection 1.4 the identity

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

was shown to hold for all acute angles. In fact it holds for all angles $\theta$, except where $\cos\theta = 0$, since it was used to *define* $\tan\theta$ for non-acute angles!

Finally, the identity

$$\sin^2\theta + \cos^2\theta = 1$$

also holds for every angle $\theta$. For example, suppose that $P$ is the point on the unit circle corresponding to an obtuse angle $\theta$. As shown in Figure 48, the line from the origin to $P$ has length 1 and it is the hypotenuse of a right-angled triangle. The length of the vertical side of the triangle is $\sin\theta$ (because $\sin\theta$ is positive). Since the $x$-coordinate of $P$ is negative, the length of the horizontal side of the triangle can be found by multiplying the $x$-coordinate of $P$ by −1. So the length of the horizontal side is $-\cos\theta$. Then, by Pythagoras' Theorem,

$$(\sin\theta)^2 + (-\cos\theta)^2 = 1^2, \quad\text{so}\quad \sin^2\theta + \cos^2\theta = 1.$$

A similar argument holds for angles in the other quadrants, so the identity is true for all values of $\theta$.

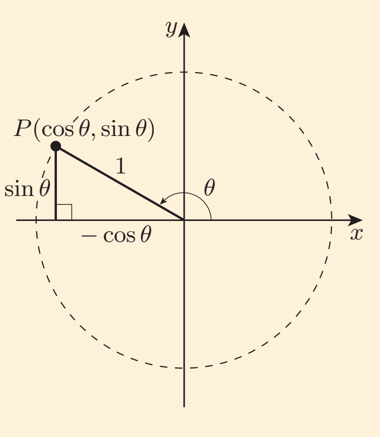



**Figure 48** A right-angled triangle

### Modelling real-world phenomena

The sine, cosine and tangent functions can be used in modelling many real-world phenomena; you will meet some examples in Unit 14. As the unit circle is used in the definition of sine and cosine, it may not surprise you that circular motion can be analysed mathematically using sines and cosines. If you go on to study higher-level modules, then you will see that this is exploited to analyse the circular motion of many types of objects, from satellites to children's roundabouts and spinning tops.

What is perhaps a little more surprising is that sines and cosines can be used to describe any periodic phenomenon such as the heights of tides, sea temperatures during the year, the motion of waves or the time when the Sun rises during the year. The video in the following activity explains how circular motion is used to predict tides. It shows how rotating circles can generate complicated periodic graphs that are similar to the graphs that show the heights of tides. The video contains an excerpt from the television series *Local Heroes* about Arthur Doodson (1890–1968), a mathematician who specialised in the prediction of tides and who designed a machine that used rotating wheels to predict the tides at ports all over the world.

> Doodson first became interested in tides when he worked at the University of Liverpool. The River Mersey has the second highest tidal range in Britain – the difference between low and high tide at Liverpool can be as much as 10 metres. This has made it harder to build modern container-handling facilities there than at other ports with less variable water levels. The largest tidal range in Britain (and the second largest in the world) is in the Bristol Channel.

Video

The video is on the DVD.

**Activity 27** *Modelling the tides*

Watch the video 'Modelling the tides' to see how the heights of tides can be predicted using a mechanical model.

So far in this section you have seen how the sine, cosine and tangent of any angle are defined, and you have also explored the graphs of these functions and seen one way they can be used. The next subsection shows you how the Sine Rule and Cosine Rule can be applied to a triangle that contains an obtuse angle.

## 3.3 Solving obtuse-angled triangles

In Section 2, you used the Sine Rule and Cosine Rule to solve acute-angled triangles. It is possible to show that these rules, along with the formula for the area of a triangle in Subsection 2.4, also hold for obtuse-angled triangles – an **obtuse-angled triangle** is one that contains an obtuse angle. These three results also hold in a right-angled triangle, but there is never any need to use them in such a triangle as it is simpler to use the trigonometric ratios sine, cosine and tangent and Pythagoras' Theorem.

The proofs are similar to the geometric proofs that you saw in Section 2 for acute-angled triangles.

This means that you can find unknown lengths and unknown angles in *any* triangle, providing that you have enough information about the other sides and angles. This is particularly useful in applications such as surveying.

When you use the Sine Rule or the Cosine Rule in a triangle with an obtuse angle, you might need to find the sine or cosine of the obtuse angle. You can use your calculator to do this in exactly the same way as for acute angles.

In Subsection 2.2 you saw that you can use the Cosine Rule to find either an unknown side or an unknown angle in a triangle. To find an unknown angle, you first use the Cosine Rule to find the value of the cosine of the angle, and then you use this value to find the angle itself. If the angle that you are trying to find happens to be obtuse, then the cosine will be negative, but you can still use the inverse cosine key on your calculator to find the angle in the usual way. For example, if you enter $\cos^{-1}(-0.5)$, then you will obtain the answer 120°.

Remember that the cosine is negative for angles between 90° and 180°.

The next activity illustrates a practical application of the Cosine Rule in an obtuse-angled triangle. If you are designing a garden, then one of the first steps is to draw a scale diagram of the boundary. To make sure that the angles and lengths on the diagram are accurate, you can use a process known as *triangulation*. This involves identifying some key points on the boundary of the garden and then measuring the distance from each of these points to two known fixed points, such as the corners of a building.

Figure 49 shows the boundary of a garden. The distances of the boundary point $B$ from the two fixed points $A$ and $C$ at the corners of the house are marked, as is the width of the house. You can use the Cosine Rule to find the angle at $A$ in the triangle $ABC$, and you can then use this angle to help you to draw an accurate scale diagram.

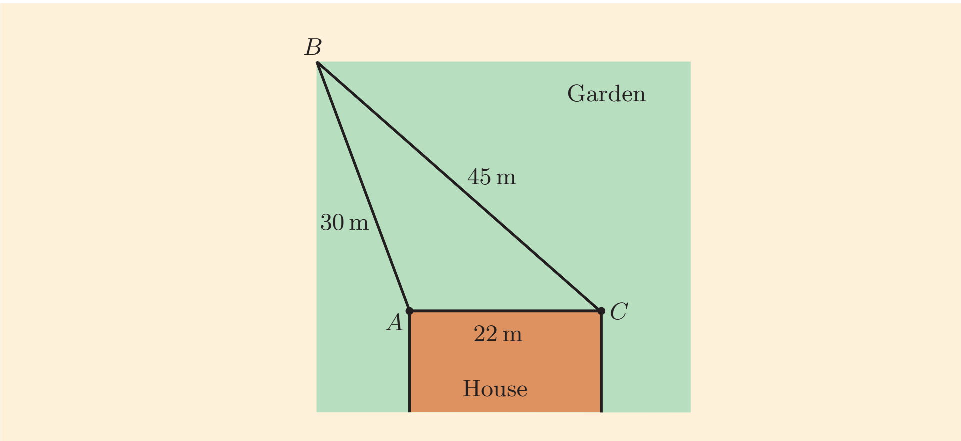



**Figure 49** Measurements in a garden

## **Activity 28** *Finding an angle using the Cosine Rule*

Use the Cosine Rule to find the value of $\cos A$ in Figure 49, and hence find the angle $A$ to the nearest degree.

It was mentioned at the end of Subsection 2.1 that you can sometimes use the Sine Rule to find unknown angles. The method is similar to the method based on the Cosine Rule: you use the Sine Rule to find the sine of the unknown angle, and then you use this value to find the angle itself. The sine of the angle is always a positive number between 0 and 1, since an angle in a triangle is always between 0° and 180°. However, there is a complication when you try to use the sine of the angle to find the angle.

Think back to the way that the sine of an angle is defined: it is the $y$-coordinate of the corresponding point $P$ on the unit circle. Figure 50(a) shows an acute angle and its corresponding point $P$. If you reflect the point $P$ in the $y$-axis, as shown in Figure 50(b), then its $y$-coordinate remains the same, but it corresponds to a different angle, namely the obtuse angle $180° - \theta$, as you can see from the diagram. So the two angles $\theta$ and $180° - \theta$ have the same sine value.

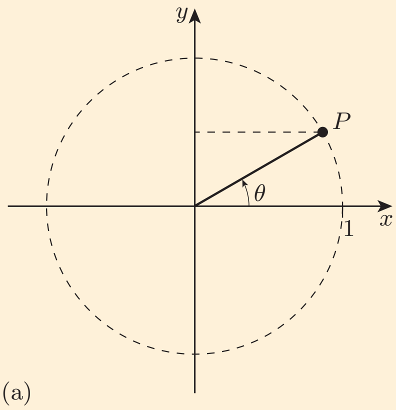



**Figure 50** Two angles that have the same sine value

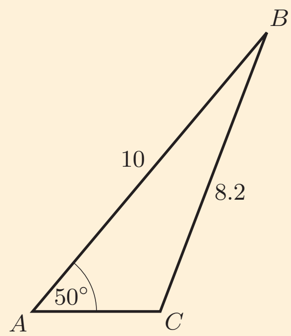
So, for example, the acute angle 30° and the obtuse angle 150° have the same sine value, because $150° = 180° - 30°$. Similarly, 60° and 120° have the same sine value, and so on. You might like to try finding the sines of some other pairs of angles of the form $\theta$ and $180° - \theta$ on your calculator.

So if you know the sine of an angle in a triangle, for example by using the Sine Rule, then there are two possibilities for the angle. It could be either the acute angle $\theta$ that you find by using the inverse sine key on your calculator, or the obtuse angle $180° - \theta$. This means that you can use the Sine Rule to find an angle only if you know enough information about the angle to be able to decide which of the two possibilities is correct. For example, you might simply be told whether the angle is acute or obtuse, as in the next example.

## Example 11 *Using the Sine Rule to find an angle*

Find the angle $C$, to the nearest degree, in the triangle below, given that it is obtuse.

The angle $C$ in the diagram *looks* obtuse, but remember that you should not assume that a geometric diagram has a property just because it looks as if it does. You can only use properties that are explicitly marked or stated.




### Solution

A side length and the opposite angle are known, so use the Sine Rule.

By the Sine Rule,

$$\frac{\sin 50^\circ}{8.2} = \frac{\sin C}{10}, \quad \text{so} \quad \sin C = \frac{10 \sin 50^\circ}{8.2} = 0.9342\ldots.$$

A calculator gives

$$\sin^{-1}(0.9342\ldots) = 69.09\ldots^\circ = 69^\circ \text{ (to the nearest degree).}$$

So the two possible values for $C$ are $69^\circ$ and $180^\circ - 69^\circ = 111^\circ$.

But $C$ is an obtuse angle, so $C = 111^\circ$ (to the nearest degree).

This solution uses the second form of the Sine Rule,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c},$$

given on page 77.

Note that you can subtract the rounded value $69^\circ$ from $180^\circ$, rather than subtracting the full-calculator-precision value and then rounding. You will get the same answer either way.

In Example 11, you saw that the Sine Rule gave two values for $C$, namely $69^\circ$ and $111^\circ$, and the acute angle was disregarded because $C$ is an obtuse angle.

In fact, two different triangles can be drawn with the measurements given in Example 11. Both triangles have an angle of $50^\circ$, an opposite side of length 8.2 and a further side of length 10. However, the other two angles of the first triangle are approximately $69^\circ$ and $61^\circ$, whereas those of the second triangle are approximately $111^\circ$ and $19^\circ$. These two triangles are shown in Figure 51.

The third angle in a triangle can be found by subtracting the two known angles from $180^\circ$.

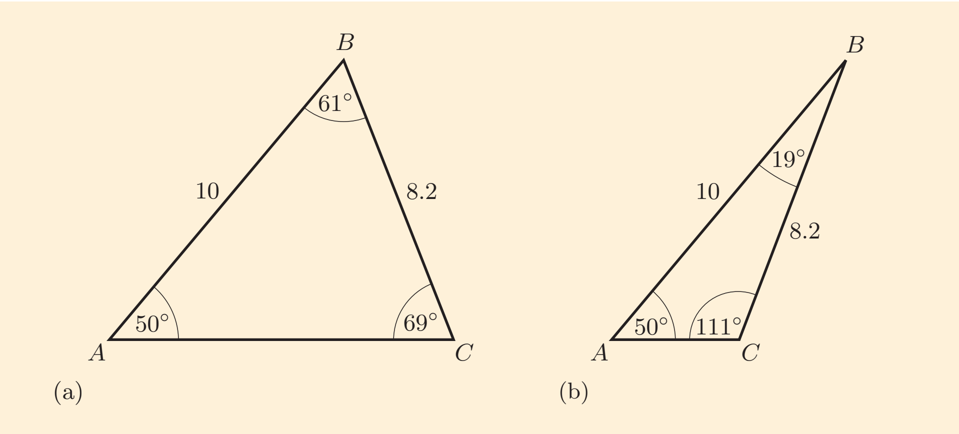
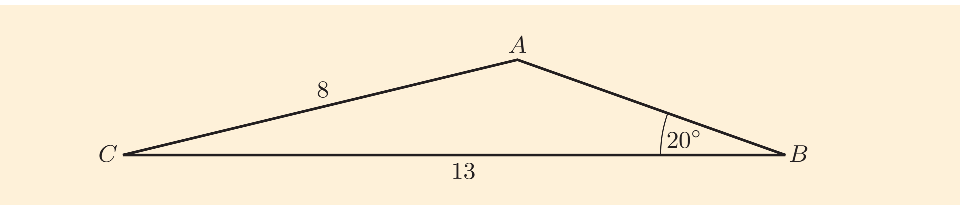



**Figure 51** Two triangles can be formed from the given measurements

This fact may not have surprised you! The initial measurements were in the order angle-side-side (ASS) and you know from Unit 8, Subsection 3.1, that this combination of an angle and two sides isn't always sufficient to determine a triangle uniquely. If you use the Sine Rule to find an unknown angle in such a triangle, then it is important to check which of the two possible values for the angle is required.

## Activity 29 *Finding an unknown angle using the Sine Rule*

(a) Use the Sine Rule to find the value of $\sin A$, to four decimal places, for the triangle below.




(b) The angle $A$ is obtuse. What is its value, to the nearest degree?

If you know the value of one of the angles of a triangle and it is at least 90°, then since the angles in a triangle add up to 180° both the other angles must be acute. So in this case, if you use the Sine Rule to find one of these other angles, then you need to find only the acute angle solution, that is, the value given by your calculator.

In this section you have seen how to define the sines, cosines and tangents of general angles measured in degrees. In the next section you will see a different method of measuring angles.

# 4 Radians

This section is concerned with another way of measuring angles, different from degrees. The number of degrees in a full turn, 360, is a matter of convention that has its roots in ancient Babylonian mathematics. This convention has persisted for so many centuries because it has worked well and there was no good reason for changing to another measurement system when applying trigonometry in practical situations such as surveying.

However, when mathematicians studied circular motion and other periodic phenomena, it became apparent that the equations involved are often simpler if you use a different unit for measuring angles, called a *radian*. In higher-level mathematics modules, angles are almost always measured in radians.

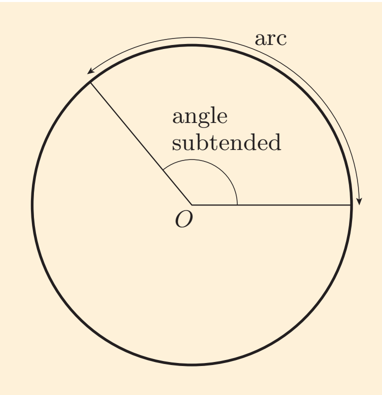



**Figure 52** The angle subtended by an arc

## 4.1 Defining a radian

You can understand what a radian is by thinking about arcs on the circumference of a circle. Figure 52 shows such an arc, and the corresponding angle at $O$, the centre of the circle. The angle at the centre is said to be **subtended** by the arc. The definition of a radian is as follows.

> ***Radians***
>
> One **radian** is the angle subtended at the centre of a circle by an arc that is the same length as the radius.

This definition is illustrated in Figure 53.

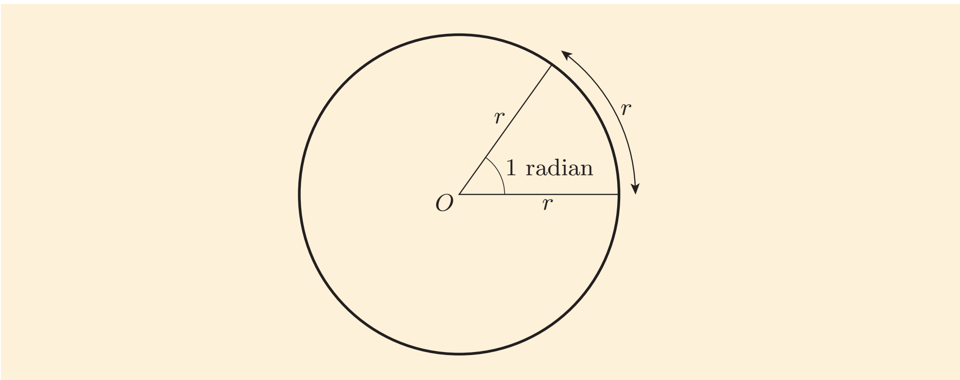



**Figure 53** An angle of one radian

From this definition, you can find the number of radians in a full turn. The circumference of the circle in Figure 53 has length $2\pi r$, and each arc of length $r$ subtends an angle of 1 radian. So the number of radians in a full turn is

$$\frac{2\pi r}{r} = 2\pi.$$

In other words, $360°$ is the same angle as $2\pi$ radians.

$$2\pi \text{ radians} = 360°.$$

This gives

$$1 \text{ radian} = \frac{360°}{2\pi} = \frac{180°}{\pi}.$$

Since

$$\frac{180}{\pi} = 57.295\ldots,$$

one radian is approximately $57°$.

Because a full turn is $2\pi$ radians, the number of radians in a simple fraction of a full turn can be conveniently expressed in terms of $\pi$. It is usual to leave these numbers in this form, rather than finding decimal approximations. For example, a half-turn is an angle of $\pi$ radians, and a quarter-turn is an angle of $\frac{1}{2}\pi$ radians, also written as $\frac{\pi}{2}$ radians or $\pi/2$ radians. Reasoning in this way, we can build up Table 3.

The expression $\pi/2$ is usually read as 'pi by two'.

**Table 3** A conversion table for common angles

| Angle in degrees | Angle in radians |
|---|---|
| $0°$ | $0$ |
| $30°$ | $\pi/6$ |
| $45°$ | $\pi/4$ |
| $60°$ | $\pi/3$ |
| $90°$ | $\pi/2$ |
| $180°$ | $\pi$ |
| $360°$ | $2\pi$ |

For example, $30°$ is one twelfth of a full turn, so $30°$ is $2\pi/12 = \pi/6$ radians.

Since

$$1 \text{ radian} = \frac{180°}{\pi},$$

the factor $180/\pi$ can be used to convert an angle measured in radians into degrees, and vice versa.

***Converting between degrees and radians***

$$\text{angle in radians} = \frac{\pi}{180} \times \text{angle in degrees},$$

$$\text{angle in degrees} = \frac{180}{\pi} \times \text{angle in radians}.$$

Here are some examples of these types of conversions.

**Example 12** *Converting between degrees and radians*

(a) Convert 270° to radians.

(b) Convert $5\pi/6$ radians to degrees.

**Solution**

(a) Applying the degrees-to-radians conversion formula gives

$$\text{angle in radians} = \frac{\pi}{180} \times 270 = \frac{3\pi}{2}.$$

So $270° = 3\pi/2$ radians.

(b) Applying the radians-to-degrees conversion formula gives

$$\text{angle in degrees} = \frac{180}{\pi} \times \frac{5\pi}{6} = 150.$$

So $5\pi/6 = 150°$.

Here are some similar questions for you to try.

**Activity 30** *Converting between degrees and radians*

(a) Convert the following angles from degrees to radians.

(i) 75° (ii) 225°

(b) Convert the following angles from radians to degrees.

(i) $3\pi/4$ radians (ii) $4\pi/5$ radians

## 4.2 Formulas involving radians

Some formulas related to circles have a simpler form if angles are measured in radians rather than degrees.

An example is the formula for the length of an arc on the circumference of a circle of radius $r$, in terms of the angle $\theta$ subtended. You know that

an arc that subtends an angle of 1 radian has length $r$.

Hence

an arc that subtends an angle of $\theta$ radians has length $r\theta$.

***Length of an arc of a circle***

$$\text{arc length} = r\theta,$$

where $r$ is the radius of the circle and $\theta$ is the angle subtended by the arc, measured in radians.

An arc of a circle is sometimes called a **circular arc**, and it can occur as part of another shape. The **centre** and **radius** of a circular arc are the centre and radius of the circle whose circumference it is part of. The next example involves a curved section of road whose edges are circular arcs.

## Example 13 *Planning a road barrier*

The diagram below shows a plan for a bend $ABCD$ in a new road. $AB$ and $CD$ are circular arcs with centre $O$. A barrier is to be placed along $CD$. What is the length of this barrier, to the nearest metre?

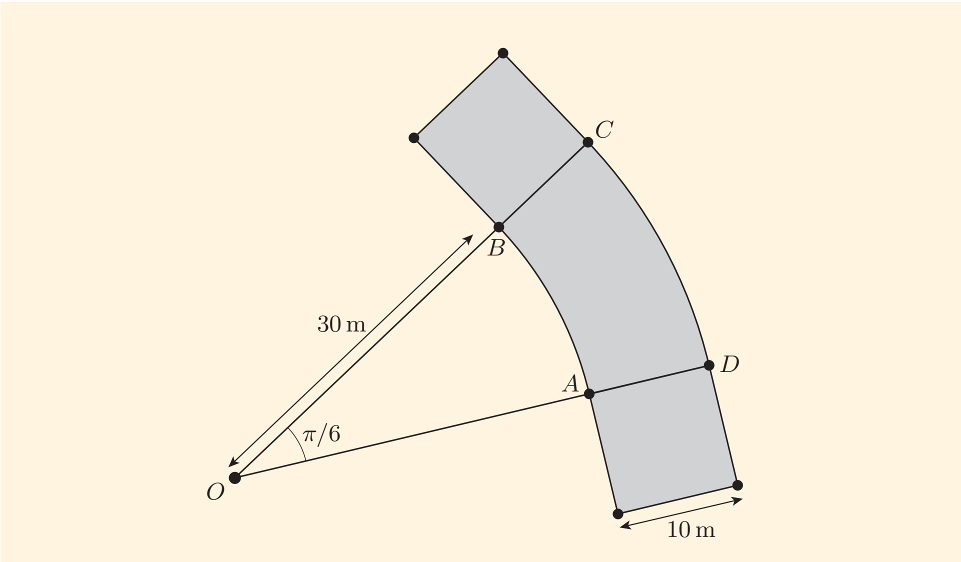



**Figure 54** The road is long, with many a winding turn

### Solution

Find the radius of the arc and the angle that it subtends at the centre of the circle.

The radius of the arc $CD$ is the length $OA + AD$, which is 40 metres.

The arc $CD$ subtends an angle of $\pi/6$ radians at the centre of the circle.

Use the formula for arc length.

So the length of the arc $CD$ is $r\theta$, where $r = 40$ and $\theta = \pi/6$.

Hence the length of the arc $CD$ in metres is

$$40 \times \frac{\pi}{6} = \frac{20\,\pi}{3} = 20.94\ldots$$

So the length of the barrier is 21 m (to the nearest metre).

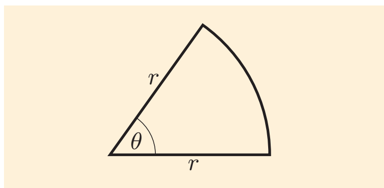



**Figure 55** A sector of a circle

Another formula that's simpler when radians are used is the formula for the area of a sector. As you saw in Unit 8, Subsection 4.3, a *sector* of a circle is the part of the circle lying between two radii, as shown in Figure 55.

Here's how to find this formula. First, the area of a sector is proportional to the angle $\theta$ of the sector. Next, if the angle of the sector is $\pi$ radians, a half-turn, then the sector is a semicircle, which has area $\frac{1}{2}\pi r^2$. That is,

an angle of $\pi$ radians gives a sector of area $\frac{1}{2}\pi r^2$.

Hence

an angle of 1 radian gives a sector of area $\frac{1}{2}r^2$,

so

an angle of $\theta$ radians gives a sector of area $\frac{1}{2}r^2\theta$.

This result is summarised overleaf.

**Area of a sector of a circle**

$$\text{area of sector} = \tfrac{1}{2}r^2\theta,$$

where $r$ is the radius of the circle and $\theta$ is the angle of the sector, measured in radians.

The following example uses the formula for the area of a sector to find the area of the road bend in Example 13.

## Example 14 *Finding the area of the bend in the road*

Calculate the area of the road bend $ABCD$ in Example 13.

### Solution

Identify the area that is required and work out how you can find it from the areas of shapes that you know how to calculate.

The area of $ABCD$ can be found by subtracting the area of the sector $OAB$ from the area of the sector $OCD$.

Use the formula for the area of a sector.

For the sector $OCD$, the radius $r$ is 40 metres and the angle is $\pi/6$. So the area in square metres of the sector $OCD$ is

$$\tfrac{1}{2} \times 40^2 \times \frac{\pi}{6} = \frac{400\pi}{3} = 418.879\ldots.$$

For the sector $OAB$, the radius $r$ is 30 metres and the angle is $\pi/6$. So the area in square metres of the sector $OAB$ is

$$\tfrac{1}{2} \times 30^2 \times \frac{\pi}{6} = 75\pi = 235.619\ldots.$$

Hence the area in square metres of the bend in the road is

$$\begin{aligned}&\text{area of sector } OCD - \text{area of sector } OAB\\ &= 418.879\ldots - 235.619\ldots\\ &= 183.259\ldots\\ &= 183 \text{ (to the nearest integer).}\end{aligned}$$

So the area of the bend is approximately 183 square metres.

The next activity involves using the formulas for arc length and the area of a sector to calculate lengths and areas in a gothic window. An example of a gothic window is shown in Figure 56.

The angles given in the activity are measured in degrees, so before you can use the formulas, the first step is to convert from degrees to radians.

## Activity 31 *Finding lengths and areas in a gothic window*

The following diagram shows a gothic window. Each of the two curves at the top is a circular arc whose centre is the lowest point of the other circular arc.

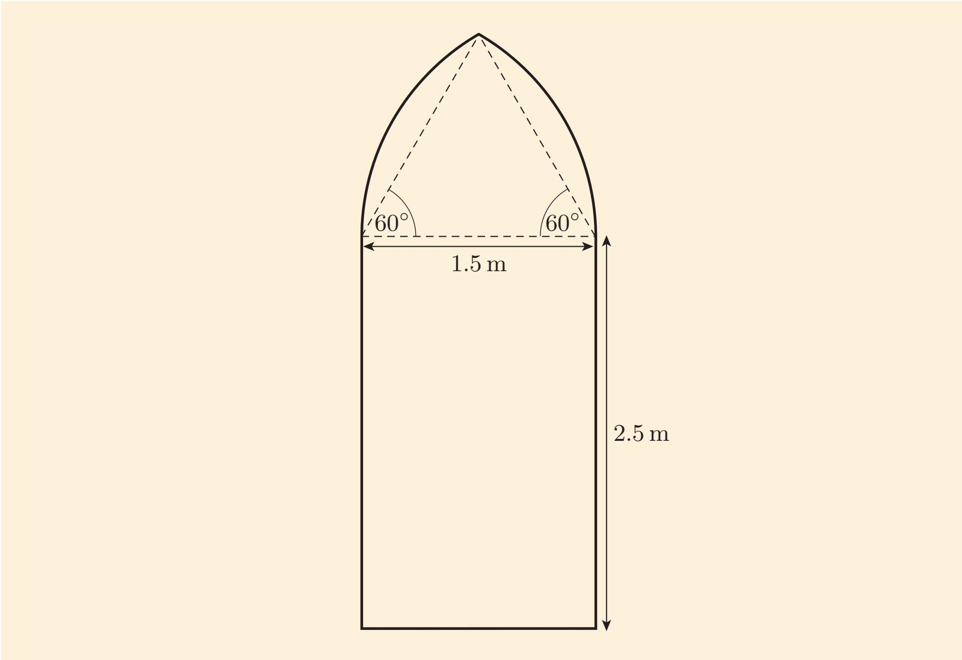



**Figure 56** A gothic window

(a) Convert 60° to radians.

(b) Calculate the length of metal edging that is required to fit around the entire perimeter of the window, in metres, to two decimal places.

(c) Calculate the area of the window, in square metres, to two decimal places, using the following steps.

  (i) Calculate the area of the triangle formed by the three dashed lines.

  (ii) Calculate the area of the sector formed by the horizontal dashed line, one slant dashed line and one of the circular arcs.

  (iii) Hence calculate the area of the window above the horizontal dashed line.

  (iv) Calculate the area of the whole window.

---

When the size of an angle is given in radians, the word 'radians' is often omitted. For example, you might say that the size of an angle is $\pi/3$, rather than $\pi/3$ radians. So if you see the size of an angle given with no degrees mentioned, then you can assume that it is measured in radians.

This convention is particularly useful when you are using trigonometry and the angles are measured in radians. For example, $\sin(\pi/3)$ means the sine of the angle that measures $\pi/3$ radians.

You can use your calculator to find the trigonometric ratios of angles measured in radians, but first it is important to check that your calculator is set to measure angles in radians rather than degrees. The next activity shows you how to do that.

In Unit 14, you will need to find the trigonometric ratios of angles measured in radians.

**Activity 32** *Using radians on your calculator*

This activity is in Subsection 3.9 of the MU123 Guide.

In Subsection 1.4 you discovered that you could work out the trigonometric ratios of some commonly-used angles without using your calculator. Table 4 shows the trigonometric ratios of these angles, with the radian measures included as well. The table also includes the angles 0° and 90°, whose trigonometric ratios you learned about in Section 3.

**Table 4** Sine, cosine and tangent of special angles

| $\theta$ in degrees | $\theta$ in radians | $\sin\theta$ | $\cos\theta$ | $\tan\theta$ |
|---|---|---|---|---|
| 0° | 0 | 0 | 1 | 0 |
| 30° | $\frac{\pi}{6}$ | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{3}}$ |
| 45° | $\frac{\pi}{4}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{\sqrt{2}}$ | 1 |
| 60° | $\frac{\pi}{3}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $\sqrt{3}$ |
| 90° | $\frac{\pi}{2}$ | 1 | 0 | – |

**Table 5** A pattern in the table

| $\theta$ | $\theta$ | $\sin\theta$ |
|---|---|---|
| 0° | 0 | $\frac{\sqrt{0}}{2} = 0$ |
| 30° | $\frac{\pi}{6}$ | $\frac{\sqrt{1}}{2} = \frac{1}{2}$ |
| 45° | $\frac{\pi}{4}$ | $\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$ |
| 60° | $\frac{\pi}{3}$ | $\frac{\sqrt{3}}{2}$ |
| 90° | $\frac{\pi}{2}$ | $\frac{\sqrt{4}}{2} = 1$ |

Fortunately there are several patterns in Table 4 that make the values easier to remember. For example, you saw in Activity 12 on page 73 how the sine and cosine values of certain angles are related. There is also a neat pattern for remembering the values of $\sin\theta$ in the table, which is given in Table 5. Remember, as well, that you can quickly work out the trigonometric ratios of 30°, 45° and 60° by drawing appropriate right-angled triangles, as you saw in Subsection 1.4.

As you continue your mathematical studies you will find that trigonometry has many applications: solving triangles is a skill that will often be useful, and the trigonometric functions will appear in a variety of topics, some of which are only distantly related to the geometry of triangles. You will meet some examples in Unit 14.

# Learning checklist

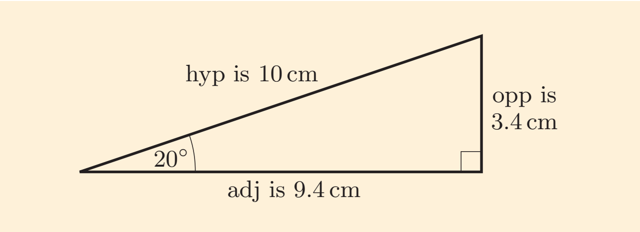
After studying this unit, you should be able to:

- define the sine, cosine and tangent of an acute angle and relate these to the sides of a right-angled triangle
- use inverse sine, inverse cosine and inverse tangent to find angles
- state and derive the sine, cosine and tangent of some special angles
- solve triangles using the Sine Rule and the Cosine Rule as appropriate
- calculate the areas of triangles from two side lengths and the included angle, or from three side lengths
- find the sine, cosine and tangent of any angle on a calculator, where these are defined
- appreciate the relationship between the sine and cosine functions, and periodic motion, in particular circular motion
- use radians as a measure of angle, and convert between radians and degrees
- calculate the lengths of arcs of circles and the areas of sectors of circles using radians.

# Solutions and comments on Activities

## Activity 1

**(a)** The marked triangle is shown below.




**(b)** The sides of the triangle measure approximately 10 cm, 3.4 cm and 9.4 cm. Using SOH CAH TOA gives

$$\sin 20^\circ = \frac{\text{opp}}{\text{hyp}} \approx \frac{3.4}{10} \approx 0.34,$$

$$\cos 20^\circ = \frac{\text{adj}}{\text{hyp}} \approx \frac{9.4}{10} \approx 0.94,$$

$$\tan 20^\circ = \frac{\text{opp}}{\text{adj}} \approx \frac{3.4}{9.4} \approx 0.36.$$

## Activity 2

The approximate value for $\tan 20^\circ$ found in Activity 1 is 0.36, so the equation is

$$\frac{x}{100} \approx 0.36$$

which gives

$$x \approx 0.36 \times 100 = 36.$$

So the height of the tree is approximately 36 m.

## Activity 4

**(a)** The unknown side is adjacent to the angle of 65°, and the given side is the hypotenuse, so the appropriate trigonometric ratio is cosine. Thus

$$\cos 65^\circ = \frac{x}{9}, \quad \text{so} \quad x = 9\cos 65^\circ.$$

Using a calculator gives $x = 3.80$ (to 3 s.f.).

**(b)** The given side is opposite the angle of 35°, and the unknown side is the hypotenuse, so the appropriate trigonometric ratio is sine. Thus

$$\sin 35^\circ = \frac{10}{x}, \quad \text{so} \quad x = \frac{10}{\sin 35^\circ}.$$

Using a calculator gives $x = 17.4$ (to 3 s.f.).

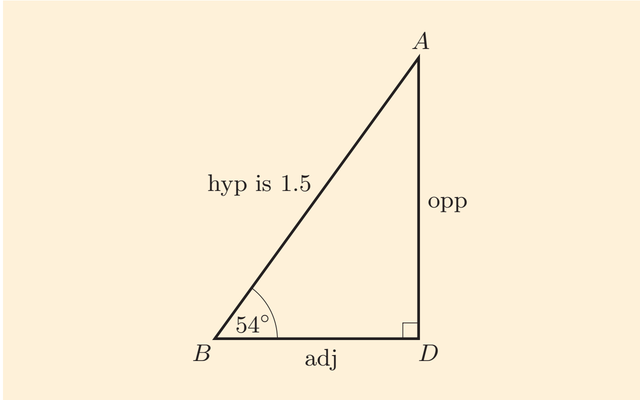
**(c)** The unknown side is opposite the angle of 48°, and the given side is adjacent to this angle, so the appropriate trigonometric ratio is tangent. Thus

$$\tan 48^\circ = \frac{x}{25}, \quad \text{so} \quad x = 25\tan 48^\circ.$$

Using a calculator gives $x = 27.76\ldots$, so the length is 27.8 m (to 3 s.f.).

## Activity 5

From the symmetry of the triangle, $BD = DC$, so $BC = 2BD$.




In the right-angled triangle $\triangle ABD$, the hypotenuse $AB$ is 1.5 m and the side adjacent to the angle 54° is $BD$. Hence

$$\cos 54^\circ = \frac{BD}{1.5}, \quad \text{so} \quad BD = 1.5\cos 54^\circ.$$

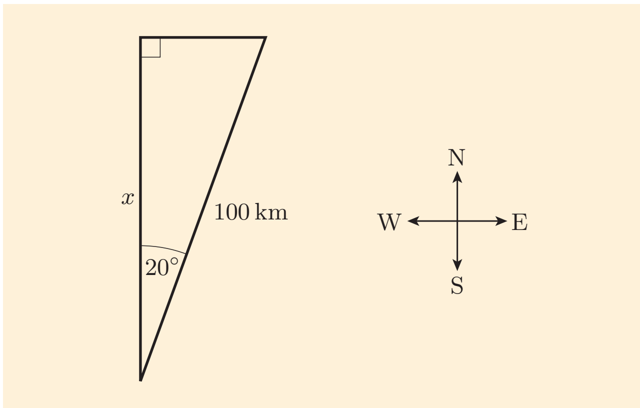
Hence

$$BC = 2 \times 1.5\cos 54^\circ = 1.7633\ldots.$$

Thus the length of $BC$ is 1.763 m, or 1763 mm (to the nearest millimetre).

## Activity 6

In the diagram below, the unknown distance is labelled $x$.




The unknown length is adjacent to the given angle. Thus

$$\cos 20^\circ = \frac{x}{100}, \quad \text{so} \quad x = 100\cos 20^\circ = 93.969\ldots.$$

Hence the plane is 94 km north of its starting position (to the nearest kilometre).

## Activity 7

**(a)** The given length 9 is opposite the angle 40° and the unknown $x$ is the length of the hypotenuse. Thus

$$\sin 40^\circ = \frac{9}{x}, \quad \text{so} \quad x = \frac{9}{\sin 40^\circ} = 14.001\ldots.$$

Hence $x = 14.0$ (to 3 s.f.).

The unknown $y$ is adjacent to the angle 40°. Thus

$$\tan 40^\circ = \frac{9}{y}, \quad \text{so} \quad y = \frac{9}{\tan 40^\circ} = 10.725\ldots.$$

Hence $y = 10.7$ (to 3 s.f.).

**(b)** The length of the hypotenuse is 10 and the unknown $x$ is opposite the angle 25°. Thus

$$\sin 25^\circ = \frac{x}{10}, \quad \text{so} \quad x = 10\sin 25^\circ = 4.226\ldots.$$

Hence $x = 4.23$ (to 3 s.f.).

The unknown $y$ is adjacent to the angle 25°. Thus

$$\cos 25^\circ = \frac{y}{10}, \quad \text{so} \quad y = 10\cos 25^\circ = 9.063\ldots.$$

Hence $y = 9.06$ (to 3 s.f.).

**(c)** The given length 11 is adjacent to the given angle 48° and the unknown $x$ is opposite this angle. Thus

$$\tan 48^\circ = \frac{x}{11}, \quad \text{so} \quad x = 11\tan 48^\circ = 12.21\ldots.$$

Hence $x = 12.2$ (to 3 s.f.).

The unknown $y$ is the hypotenuse. Thus

$$\cos 48^\circ = \frac{11}{y}, \quad \text{so} \quad y = \frac{11}{\cos 48^\circ} = 16.43\ldots.$$

Hence $y = 16.4$ (to 3 s.f.).

**(d)** The given length 8 is opposite the given angle 63° and the unknown $x$ is adjacent to this angle. Thus

$$\tan 63^\circ = \frac{8}{x}, \quad \text{so} \quad x = \frac{8}{\tan 63^\circ} = 4.076\ldots.$$

Hence $x = 4.08$ (to 3 s.f.).

The unknown $y$ is the hypotenuse. Thus

$$\sin 63^\circ = \frac{8}{y}, \quad \text{so} \quad y = \frac{8}{\sin 63^\circ} = 8.978\ldots.$$

Hence $y = 8.98$ (to 3 s.f.).

## Activity 9

**(a)** In this triangle, you know the length of the side opposite the angle $\alpha$ and the length of the hypotenuse. So

$$\sin\alpha = \frac{10}{15} = \frac{2}{3}.$$

Thus $\alpha$ is the acute angle whose sine is $\frac{2}{3}$. A calculator gives $\sin^{-1}(\frac{2}{3}) = 41.8\ldots^\circ$, so the unknown angle is 42° (to the nearest degree).

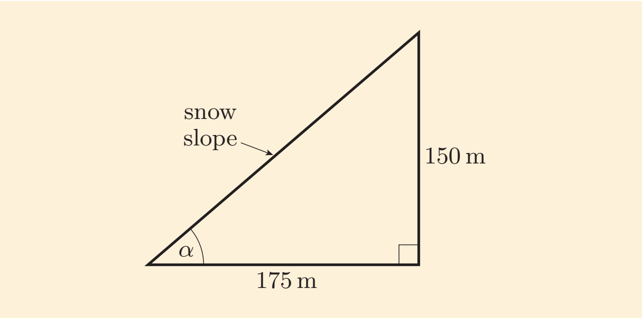
**(b)** In this triangle, you know the length of the side opposite the angle $\beta$ and the length of the side adjacent to it. So

$$\tan\beta = \frac{9}{8}.$$

Thus $\beta$ is the acute angle whose tangent is $\frac{9}{8}$. A calculator gives $\tan^{-1}(\frac{9}{8}) = 48.3\ldots^\circ$, so the unknown angle is 48° (to the nearest degree).

## Activity 10

Let $\alpha$ represent the angle that the snow slope makes with the horizontal.




From the diagram, $\tan\alpha = \frac{150}{175} = \frac{6}{7}$.

So $\alpha = \tan^{-1}(\frac{6}{7}) = 40.6\ldots^\circ$.

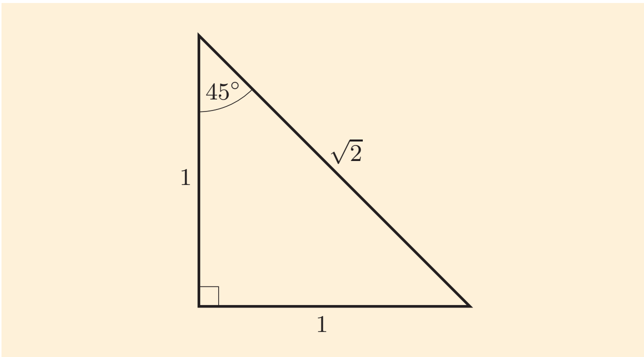
Since $35^\circ < \alpha < 45^\circ$, there is a risk of an avalanche.

(Note that many other factors may contribute to the risk of an avalanche!)

## Activity 11

The length of the hypotenuse, $x$, can be calculated by applying Pythagoras' Theorem:

$$x^2 = 1^2 + 1^2 = 2, \quad \text{so} \quad x = \sqrt{2}.$$




So the hypotenuse has length $\sqrt{2}$ and the adjacent and opposite sides have length 1. Thus

$$\sin 45^\circ = \frac{1}{\sqrt{2}}, \quad \cos 45^\circ = \frac{1}{\sqrt{2}}, \quad \tan 45^\circ = \frac{1}{1} = 1.$$

## Activity 12

**(a)** In this triangle,

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{a}{c}, \quad \cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{b}{c}.$$

Also,

$$\sin(90^\circ - \theta) = \frac{\text{opp}}{\text{hyp}} = \frac{b}{c}, \quad \cos(90^\circ - \theta) = \frac{\text{adj}}{\text{hyp}} = \frac{a}{c}.$$

**(b)** By the equations in part (a),

$$\cos\theta = \frac{b}{c} = \sin(90^\circ - \theta)$$

and

$$\sin\theta = \frac{a}{c} = \cos(90^\circ - \theta).$$

## Activity 14

The side length 7 m is opposite the angle of 55°, and the unknown side length $x$ is opposite the angle of 60°. By the Sine Rule,

$$\frac{7}{\sin 55^\circ} = \frac{x}{\sin 60^\circ},$$

so

$$x = \frac{7\sin 60^\circ}{\sin 55^\circ} = 7.400\ldots.$$

Thus $x = 7.40\,\text{m}$ (to 3 s.f.).

## Activity 15

The unknown side length $x$ is opposite the angle of 55°, and this angle is between two known side lengths. So we can apply the Cosine Rule to the triangle to give

$$x^2 = 6^2 + 10^2 - 2 \times 6 \times 10\cos 55^\circ = 67.170\ldots.$$

Thus $x = \sqrt{67.170\ldots} = 8.19\ldots = 8.2$ (to 2 s.f.).

## Activity 16

We first calculate $\angle B$ using the Cosine Rule. The side opposite $\angle B$ has length 4, and this appears on the left-hand side of the equation, with the other two lengths on the right-hand side:

$$4^2 = 3^2 + 4^2 - 2 \times 3 \times 4\cos B,$$

so

$$16 = 25 - 24\cos B.$$

Hence

$$\cos B = \frac{25 - 16}{24} = \frac{9}{24} = 0.375.$$

A calculator gives $\cos^{-1}(0.375) = 67.975\ldots$, so $\angle B = 68^\circ$ to the nearest degree.

As the triangle is isosceles, $\angle A = \angle B$. So $\angle A = 68^\circ$ to the nearest degree.

The third angle can be found by using the fact that the angles of a triangle add up to 180°. Using the unrounded value of $\angle B$, we obtain the following value for $\angle C$:

$$180 - 2 \times 67.975\ldots = 44.048\ldots.$$

So $\angle C = 44^\circ$ to the nearest degree.

## Activity 17

**(a)** This is a right-angled triangle, so go down the left-hand side of the decision tree. The problem of finding $x$ involves only lengths, so the best method to use is Pythagoras' Theorem.

To find $\theta$ use a trigonometric ratio – in this case you can find $\theta$ from the equation $\tan\theta = 6/8$.

**(b)** This is not a right-angled triangle, so follow the right-hand side of the decision tree. The known side length, 6, is not opposite a known angle, but you can calculate the angle $\theta$ to be $180^\circ - 55^\circ - 60^\circ = 65^\circ$. (This is the case indicated in brackets in the decision tree.)

Now you know a side length, 6, opposite a known angle, 65°, so you can use the Sine Rule to calculate $x$.

**(c)** Again, this is not a right-angled triangle, so follow the right-hand side of the decision tree. No angles are given, so the Sine Rule does not apply. But you know three side lengths, so you can find $\theta$ by using the Cosine Rule.

## Activity 18

**(a)** The unknown length $x$ is opposite a known angle, but the given length, the baseline 50 m, is opposite an unknown angle. So the first task is to calculate the angle opposite the baseline. Since the angles in a triangle add up to 180°, the angle opposite the baseline is $180^\circ - 78.5^\circ - 82.4^\circ = 19.1^\circ$.

Now use the Sine Rule:

$$\frac{50}{\sin 19.1^\circ} = \frac{x}{\sin 82.4^\circ},$$

so

$$x = \frac{50\sin 82.4^\circ}{\sin 19.1^\circ}.$$

Using a calculator gives $x = 151.46\,\text{m}$ (to the nearest centimetre).

**(b)** Proceeding in a similar manner on the second day, start by calculating the third angle in the triangle, which is $180^\circ - 79.0^\circ - 81.9^\circ = 19.1^\circ$.

Now use the Sine Rule:

$$\frac{50}{\sin 19.1^\circ} = \frac{y}{\sin 81.9^\circ},$$

so

$$y = \frac{50\sin 81.9^\circ}{\sin 19.1^\circ}.$$

Using a calculator gives $y = 151.28\,\text{m}$ (to the nearest centimetre).

## Activity 19

The angle opposite the unknown length $z$ is the difference between the angles measured from the left-hand end of the baseline on day 1 and day 2, that is, $79° - 78.5° = 0.5°$.

By the Cosine Rule,

$$\begin{aligned} z^2 &= 151.46^2 + 151.28^2 \\ &\quad - 2 \times 151.46 \times 151.28 \cos 0.5° \\ &= 1.777\ldots. \end{aligned}$$

So $z = \sqrt{1.777\ldots} = 1.33\ldots$. Hence $z = 1.3\,\text{m}$ (to 2 s.f.).

## Activity 20

**(a)** By the area formula, the area of the triangle in $\text{km}^2$ is

$$\tfrac{1}{2} \times 8 \times 10 \sin 40° = 25.71\ldots.$$

Thus the area is $25.7\,\text{km}^2$ (to 3 s.f.).

**(b)** By the area formula, the area of the triangle in $\text{cm}^2$ is

$$\tfrac{1}{2} \times 6 \times 7 \sin 60° = 18.18\ldots.$$

Thus the area is $18.2\,\text{cm}^2$ (to 3 s.f.).

## Activity 21

**(a)** Since the side length of the equilateral triangle is 2 and each angle is 60°, the area is

$$\tfrac{1}{2} \times 2 \times 2 \sin 60° = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}.$$

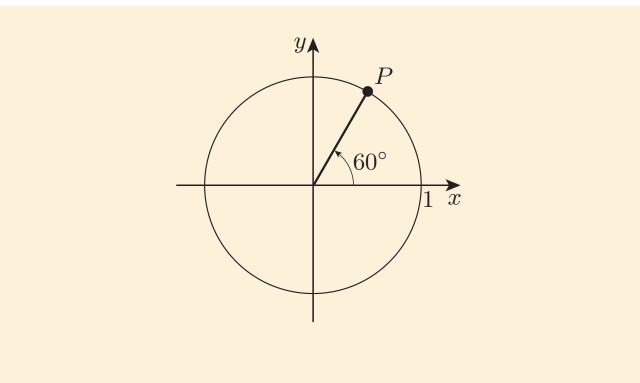
**(b)** Each side has length 2, so the semi-perimeter is $s = \frac{1}{2}(2+2+2) = 3$. By Heron's Formula, the area is

$$\begin{aligned} &\sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{3(3-2)(3-2)(3-2)} \\ &= \sqrt{3}, \end{aligned}$$

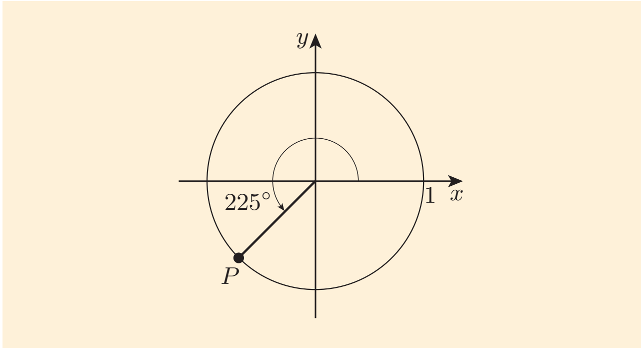
as in part (a).

## Activity 22

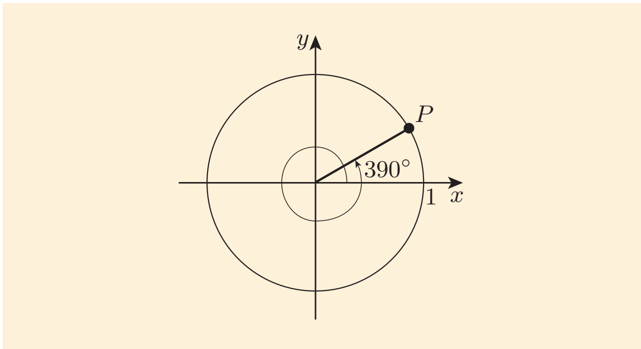
**(a)** $P$ lies in the first quadrant.



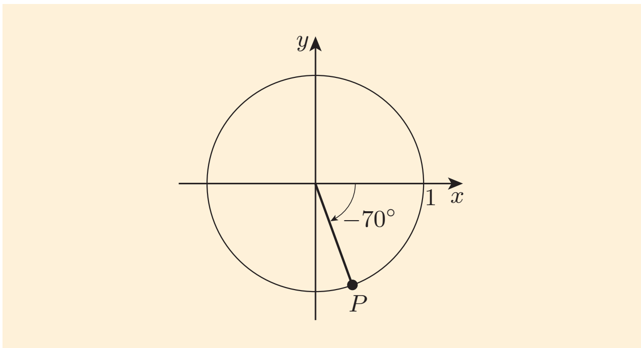

**(b)** $P$ lies in the third quadrant.




**(c)** $P$ lies in the first quadrant.

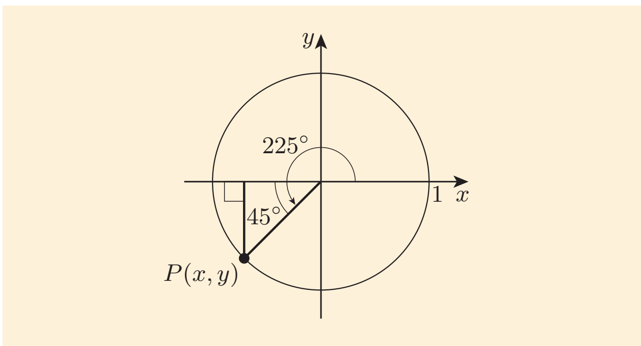



**(d)** $P$ lies in the fourth quadrant.




## Activity 23

**(a)**




The hypotenuse of the right-angled triangle in the diagram has length 1, so

$$\sin 45° = \frac{\text{opp}}{1} \quad \text{and} \quad \cos 45° = \frac{\text{adj}}{1}.$$

Hence the side opposite the angle of 45° has length

$$\sin 45° = \frac{1}{\sqrt{2}},$$

and the side adjacent to the angle of 45° has length

$$\cos 45° = \frac{1}{\sqrt{2}}.$$

(You can find the values of $\sin 45°$ and $\cos 45°$ either by using your calculator or from Table 1 on page 73.)

Therefore the coordinates of the point $P$ are

$$\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right).$$

So, by the definitions of the sine and cosine of a general angle,

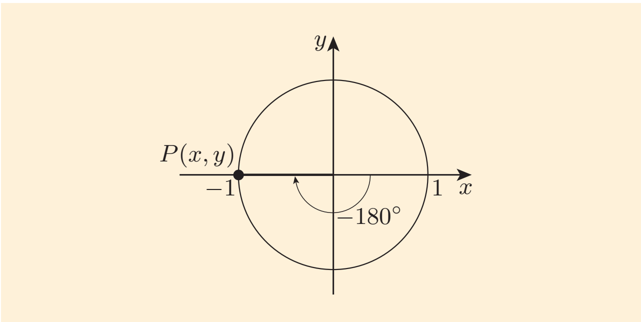
$$\cos 225° = -\frac{1}{\sqrt{2}} = -0.7071 \text{ (to 4 d.p.)}$$

and

$$\sin 225° = -\frac{1}{\sqrt{2}} = -0.7071 \text{ (to 4 d.p.)}.$$

These values agree with those given by a calculator.

**(b)**




Since $P$ has coordinates $(-1, 0)$,

$$\cos(-180°) = -1 \quad \text{and} \quad \sin(-180°) = 0.$$

These values agree with those given by a calculator.

## Activity 24

By the solution to Activity 23(a),

$$\tan 225° = \frac{\sin 225°}{\cos 225°} = \frac{-\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}} = 1.$$

By the solution to Activity 23(b),

$$\tan(-180°) = \frac{\sin(-180°)}{\cos(-180°)} = \frac{0}{-1} = 0.$$

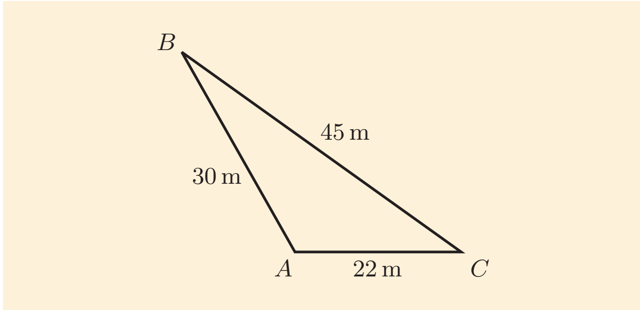
## Activity 26

The asymptotes of the tangent graph occur when $\cos\theta = 0$.

So for the portion of the tangent graph between 360° and 720°, the asymptotes of this graph are $\theta = 450°$ and $\theta = 630°$.

## Activity 28

The dimensions of the triangle are shown below.




By the Cosine Rule,

$$a^2 = b^2 + c^2 - 2bc\cos A,$$

so

$$\begin{aligned}\cos A &= \frac{b^2 + c^2 - a^2}{2bc}\\ &= \frac{22^2 + 30^2 - 45^2}{2 \times 22 \times 30}\\ &= -0.4856\ldots.\end{aligned}$$

Using a calculator gives

$$\cos^{-1}(-0.4856\ldots) = 119.05\ldots°.$$

So $A = 119°$ (to the nearest degree).

## Activity 29

**(a)** By the Sine Rule,

$$\frac{\sin A}{13} = \frac{\sin 20°}{8}.$$

Thus

$$\sin A = \frac{13 \sin 20°}{8} = 0.555\,78\ldots,$$

so $\sin A = 0.5558$ (to 4 d.p.).

**(b)** Using a calculator gives

$$\sin^{-1}(0.555\,78\ldots) = 33.76\ldots°.$$

So, to the nearest degree, the two possible angles are 34° and $180° - 34° = 146°$.

Since $A$ is obtuse, the value of $A$ is 146°.

## Activity 30

**(a) (i)** Applying the degrees-to-radians conversion formula gives

$$\text{angle in radians} = \frac{\pi}{180} \times 75 = \frac{5\pi}{12}.$$

So $75° = 5\pi/12$ radians.

**(ii)** Applying the degrees-to-radians conversion formula gives

$$\text{angle in radians} = \frac{\pi}{180} \times 225 = \frac{5\pi}{4}.$$

So $225° = 5\pi/4$ radians.

**(b)** **(i)** Applying the radians-to-degrees conversion formula gives

$$\text{angle in degrees} = \frac{180}{\pi} \times \frac{3\pi}{4} = 135.$$

So $3\pi/4$ radians $= 135°$.

**(ii)** Applying the radians-to-degrees conversion formula gives

$$\text{angle in degrees} = \frac{180}{\pi} \times \frac{4\pi}{5} = 144.$$

So $4\pi/5$ radians $= 144°$.

## Activity 31

**(a)** Applying the degrees-to-radians conversion formula gives

$$\text{angle in radians} = \frac{\pi}{180} \times 60 = \frac{\pi}{3}.$$

So $60° = \pi/3$ radians.

(Alternatively, you could have looked up the angle $60°$ in Table 3 on page 103.)

**(b)** The length of each of the two arcs in the diagram is given by $r\theta$, where $r = 1.5$ and $\theta = \pi/3$. So the length of each arc in metres is

$$1.5 \times \pi/3 = 0.5\pi.$$

Hence the total length of edging required is

$$\begin{aligned} 2 \times 0.5\pi + 2 \times 2.5 + 1.5 &= \pi + 6.5 \\ &= 9.64\,\text{m (to 2 d.p.)}. \end{aligned}$$

**(c)** **(i)** The area of the triangle is given by the formula

$$\text{area} = \tfrac{1}{2}ab\sin\theta,$$

with $a = b = 1.5$ and $\theta = 60°$. So it is

$$\tfrac{1}{2} \times 1.5^2 \times \frac{\sqrt{3}}{2} = \frac{9\sqrt{3}}{16} = 0.974\ldots\,\text{m}^2.$$

**(ii)** The area of the sector is given by the formula

$$\text{area} = \tfrac{1}{2}r^2\theta,$$

with $r = 1.5$ and $\theta = \pi/3$. So it is

$$\tfrac{1}{2} \times 1.5^2 \times \frac{\pi}{3} = \frac{3\pi}{8} = 1.178\ldots\,\text{m}^2.$$

**(iii)** There are two sectors visible in the diagram of the window, and their overlap is the triangle formed by the dashed lines. So the area of the window above the horizontal dashed line is

$$\begin{aligned} &2 \times \text{area of sector} - \text{area of triangle} \\ &= 2 \times 1.178\ldots - 0.974\ldots \\ &= 1.381\ldots\,\text{m}^2. \end{aligned}$$

(Alternatively, you could subtract the area of the triangle from the area of one sector to find the area of one of the two thin segments at the top of the window. Then you could calculate the area of the window above the horizontal dashed line by adding twice the area of the segment to the area of the triangle.)

**(iv)** The area of the window below the horizontal dashed line is

$$1.5 \times 2.5 = 3.75\,\text{m}^2,$$

so the area of the whole window is

$$1.381\ldots + 3.75 = 5.13\,\text{m}^2 \text{ (to 2 d.p.)}.$$